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# Matching procedure for the sixth Painlevé equation

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## Abstract

In the framework of the isomonodromy deformation method, we present a constructive procedure to obtain the critical behaviour of Painlevé VI transcendents and solve the connection problem. This procedure yields two- and one-parameter families of solutions, including trigonometric and logarithmic behaviours, and three classes of solutions with Taylor expansion at a critical point.

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## 1. Introduction

The sixth Painlevé equation is

$$\frac{d^2y}{dx^2} = \frac{1}{2} \left[ \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right] \left( \frac{dy}{dx} \right)^2 - \left[ \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right] \frac{dy}{dx} + \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left[ \alpha + \beta \frac{x}{y^2} + \gamma \frac{x-1}{(y-1)^2} + \delta \frac{x(x-1)}{(y-x)^2} \right] \quad (\text{PVI}).$$

The generic solution has essential singularities and/or branch points in  $0, 1, \infty$ . Its behaviour at these points will be called *critical*. The other singularities, which depend on the initial conditions, are poles. A solution of PVI can be analytically continued to a meromorphic function on the universal covering of  $\mathbf{P}^1 \setminus \{0, 1, \infty\}$ . For generic values of the integration constants and of the parameters  $\alpha, \beta, \gamma, \delta$ , it cannot be expressed via elementary or classical transcendental functions. For this reason, it is called a *Painlevé transcendent*.

Solving (PVI) means: (i) determine the critical behaviour of the transcendents at the *critical points*  $x = 0, 1, \infty$ . Such a behaviour must depend on two integration constants. (ii) Solve the *connection problem*, namely, find the relation between couples of integration constants at  $x = 0, 1, \infty$ .

In this paper, we present a procedure to compute the first leading terms of the critical behaviour at  $x = 0$ . The procedure, which in the title and in the paper is called *matching*, is

essentially the isomonodromy deformation method. The reason for our terminology is that we make particular use of the matching between local solutions of two different reductions of the linear system of ODE associated with (PVI) by the isomonodromy deformation theory. This matching allows us to obtain the leading term(s) of the asymptotic behaviour of a corresponding Painlevé transcendent  $y(x)$ . In this sense, we say that our approach is constructive. Namely, we do not assume any behaviour of  $y(x)$ ; rather, we obtain it from the matching condition. This differs from other authors' approach, who start by assuming a given asymptotics for  $y(x)$  and then compute the corresponding monodromy data (and so they solve the connection problem). This kind of approach was successfully used for some of the Painlevé equations and allowed many progresses. Our approach is developed to tackle with the cases when we do not know—or we are not able to guess—the asymptotic behaviour. In the case of (PVI), we may say that most of the solutions are known. But for some points in the space of monodromy data, we still do not know the corresponding critical behaviours. Our work is motivated by the need to explore these remaining cases.

Once the local matching is done, as a second step, we proceed with a global description of the solutions of the associate linear system of ODE, in order to compute its *monodromy data*. These are the monodromy data associated with the solution  $y(x)$ , of which the asymptotic behaviour has been obtained by the precedent step. Again, this computation is done by a (global) matching, among solutions of the two reduced systems and that of the original one. This approach, for (PVI), first appeared in [16]. It is the main powerful point of the isomonodromy deformation method. The monodromy data are computed in terms of the coefficients of the linear system of ODE, which are elementary functions of the parameters (namely, the integration constants) appearing in the leading term of the asymptotic behaviour of  $y(x)$ . The inversion of the formulae expressing the monodromy data gives the leading term of  $y(x)$  in terms of the monodromy data.

The procedure can be repeated at the other singularities  $x = 1, \infty$ . In the case of (PVI),  $x = 0, 1, \infty$  are equivalent by symmetry transformations. These facts allow us to solve the connection problem<sup>1</sup>.

A characteristic of this paper is that we obtain non-Fuchsian isomonodromic systems when we reduce the linear system of ODE associated with (PVI). This point will be clearly explained later. A matching procedure, to obtain asymptotic behaviours, has been used for the fifth Painlevé equation by Andreev and Kitaev [2]. An analogous scheme is used in [1].

As a result of the matching procedure, we obtain (1) a two-parameter family of solutions, of the type found by Jimbo [16]. Besides, we show that there are solutions with trigonometric behaviour. (2) One-parameter families of solutions, including a class of logarithmic solutions. (3) The solutions which admit a Taylor expansion at  $x = 0$ . (4) Then, we compute the corresponding monodromy data. In virtue of the symmetries of (PVI) (birational transformations of  $(x, y(x))$ ), it can be shown that the solutions with Taylor expansion at  $x = 0$ , obtained by the matching procedure, are the representatives of three equivalent classes, which include all the solutions admitting a Taylor expansion at a critical point.

<sup>1</sup> For reasons of space, we limit ourselves to the computation of monodromy data, without explaining how the connection problem is practically solved once the monodromy data are computed and how the analytic continuation is done. We refer the reader to [5, 8, 9, 11, 16]. The behaviours at  $x = 1$  and  $x = \infty$ , and their dependence on the monodromy data, are deduced from the behaviour at  $x = 0$  by symmetry transformations. PVI is invariant for the change of variables  $y(x) = 1 - \tilde{y}(t)$ ,  $x = 1 - t$  and simultaneous permutation of  $\theta_0, \theta_1$ . This means that  $y(x)$  solves PVI if and only if  $\tilde{y}(t)$  solves PVI with permuted parameters and independent variable  $t$ . Similarly, PVI is invariant for  $y(x) = 1/\tilde{y}(t)$ ,  $x = 1/t$  and simultaneous permutation of  $\theta_\infty, \theta_0$ . It is invariant for  $y(x) = (\tilde{y}(t) - t)/(1 - t)$ ,  $x = t/(t - 1)$  and simultaneous permutation of  $\theta_0, \theta_x$ . By composing the third, first and again third symmetries, we get  $y(x) = \tilde{y}(t)/t$ ,  $t = 1/x$  with the permutation of  $\theta_1, \theta_x$ . Therefore, the critical points  $0, 1, \infty$  are equivalent.

As already stressed, the purpose of this paper is to present a constructive procedure to solve (PVI) and to show its effectiveness by both reproducing known solutions and by finding new ones. It is not our purpose here to discuss the problem of characterization and classification of all the solutions of (PVI) in terms of the monodromy data of the associated linear system. Nevertheless, we show that the matching procedure is effective to produce new solutions, associated with monodromy data for which the connection problem has not been studied so far. Therefore, it is a tool to study the classification problem. This will be done in another paper.

**Part I. Matching procedure and results**

**2. Matching procedure**

PVI is the isomonodromy deformation equation of a Fuchsian system of differential equations [17]:

$$\frac{d\Psi}{d\lambda} = A(\lambda, x, \theta)\Psi, \quad A(\lambda, x, \theta) := \left[ \frac{A_0(x, \theta)}{\lambda} + \frac{A_x(x, \theta)}{\lambda - x} + \frac{A_1(x, \theta)}{\lambda - 1} \right], \quad \lambda \in \mathbf{C}. \tag{1}$$

The  $2 \times 2$  matrices  $A_i(x, \theta)$  depend on  $x$  in such a way that the monodromy of a fundamental solution  $\Psi(\lambda, x)$  does not change for small deformations of  $x$ . They also depend on the parameters  $\alpha, \beta, \gamma, \delta$  of PVI through more elementary parameters  $\theta = (\theta_0, \theta_x, \theta_1, \theta_\infty)$  according to the following relations:

$$\begin{aligned} -A_\infty &:= A_0 + A_1 + A_x = -\frac{\theta_\infty}{2}\sigma_3, \quad \theta_\infty \neq 0; & \text{eigenvalues } (A_i) &= \pm \frac{1}{2}\theta_i, \quad i = 0, 1, x; \\ \alpha &= \frac{1}{2}(\theta_\infty - 1)^2, & -\beta &= \frac{1}{2}\theta_0^2, & \gamma &= \frac{1}{2}\theta_1^2, & \left(\frac{1}{2} - \delta\right) &= \frac{1}{2}\theta_x^2. \end{aligned} \tag{2}$$

Here,  $\sigma_3$  is the Pauli matrix. The equations of monodromy-preserving deformation (Schlesinger equations) can be written in Hamiltonian form and reduce to PVI, being the transcendent  $y(x)$  solution of  $A(y(x), x, \theta)_{1,2} = 0$ . Namely,

$$y(x) = \frac{x(A_0)_{12}}{x[(A_0)_{12} + (A_1)_{12}] - (A_1)_{12}}. \tag{3}$$

The matrices  $A_i(x, \theta), i = 0, x, 1$ , depend on  $y(x), \frac{dy(x)}{dx}$  and  $\int y(x)$  through rational functions, which are given in [17]. In short, we will write  $A_i = A_i(x)$ .

The product of the monodromy matrices  $M_0, M_x, M_1$  of a fundamental matrix solution  $\Psi$  at  $\lambda = 0, x, 1$ , respectively, is equal to the monodromy at  $\lambda = \infty$ . The order of the products depends on the choice of a basis of loops. As a consequence, the following relation must hold:

$$\begin{aligned} \cos(\pi\theta_0) \operatorname{tr}(M_1 M_x) + \cos(\pi\theta_1) \operatorname{tr}(M_0 M_x) + \cos(\pi\theta_x) \operatorname{tr}(M_1 M_0) \\ = 2 \cos(\pi\theta_\infty) + 4 \cos(\pi\theta_1) \cos(\pi\theta_0) \cos(\pi\theta_x). \end{aligned}$$

*2.1. Leading terms of  $y(x)$  as a result of matching*

We present the constructive procedure to obtain the leading terms of a solution  $y(x)$ , when  $x \rightarrow 0$ . This procedure has been used for the fifth Painlevé equation by Andreev and Kitaev [2]. An analogous scheme is used in [1].

Since we are considering  $x \rightarrow 0$ , we divide the  $\lambda$ -plane into two domains. The ‘outside’ domain is defined for  $\lambda$  sufficiently big:

$$|\lambda| \geq |x|^{\delta_{\text{OUT}}}, \quad \delta_{\text{OUT}} > 0. \quad (4)$$

Therefore, (1) can be written as

$$\frac{d\Psi}{d\lambda} = \left[ \frac{A_0 + A_x}{\lambda} + \frac{A_x}{\lambda} \sum_{n=1}^{\infty} \left(\frac{x}{\lambda}\right)^n + \frac{A_1}{\lambda - 1} \right] \Psi. \quad (5)$$

The ‘inside’ domain is defined for  $\lambda$  comparable with  $x$ , namely,

$$|\lambda| \leq |x|^{\delta_{\text{IN}}}, \quad \delta_{\text{IN}} > 0. \quad (6)$$

Therefore,  $\lambda \rightarrow 0$  as  $x \rightarrow 0$ , and we rewrite (1) as

$$\frac{d\Psi}{d\lambda} = \left[ \frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} - A_1 \sum_{n=0}^{\infty} \lambda^n \right] \Psi. \quad (7)$$

If the behaviour of  $A_0(x)$ ,  $A_1(x)$  and  $A_x(x)$  is sufficiently good, we expect that the higher order terms in the series of (5) and (7) are small corrections which can be neglected when  $x \rightarrow 0$ . If this is the case, (5) and (7) reduce respectively to

$$\frac{d\Psi_{\text{OUT}}}{d\lambda} = \left[ \frac{A_0 + A_x}{\lambda} + \frac{A_x}{\lambda} \sum_{n=1}^{N_{\text{OUT}}} \left(\frac{x}{\lambda}\right)^n + \frac{A_1}{\lambda - 1} \right] \Psi_{\text{OUT}}, \quad (8)$$

$$\frac{d\Psi_{\text{IN}}}{d\lambda} = \left[ \frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} - A_1 \sum_{n=0}^{N_{\text{IN}}} \lambda^n \right] \Psi_{\text{IN}}, \quad (9)$$

where  $N_{\text{IN}}$ ,  $N_{\text{OUT}}$  are the suitable integers. The simplest reduction is to Fuchsian systems:

$$\frac{d\Psi_{\text{OUT}}}{d\lambda} = \left[ \frac{A_0 + A_x}{\lambda} + \frac{A_1}{\lambda - 1} \right] \Psi_{\text{OUT}}, \quad (10)$$

$$\frac{d\Psi_{\text{IN}}}{d\lambda} = \left[ \frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} \right] \Psi_{\text{IN}}. \quad (11)$$

It is a new feature of this paper that we will use reduced non-Fuchsian systems. In the literature, the Fuchsian reduction has been privileged, but we show that in some relevant cases it cannot be used, being the non-Fuchsian reduction necessary.

Generally speaking, we can parameterize the elements of  $A_0 + A_x$  and  $A_1$  of (10) in terms of  $\theta_1$ , the eigenvalues of  $A_0 + A_x$  and the eigenvalues  $\theta_{\infty}$  of  $A_0 + A_x + A_1$ . We also need an additional unknown function of  $x$ . In the same way, we can explicitly parameterize the elements of  $A_0$  and  $A_x$  in (11) in terms of  $\theta_0$ ,  $\theta_x$ , the eigenvalues of  $A_0 + A_x$  and another additional unknown function of  $x$ . When the reductions (8) and (9) are non-Fuchsian, particular care must be paid. This will be explained case by case in the paper. Our purpose is to find the leading term of the unknown functions when  $x \rightarrow 0$ , in order to determine the critical behaviour of  $A_0(x)$ ,  $A_1(x)$ ,  $A_x(x)$  and (3).

The leading term can be obtained as a result of two facts:

- (i) Systems (8) and (9) are isomonodromic. This imposes constraints on the form of the unknown functions. Typically, one of them must be constant.

(ii) Two fundamental matrix solutions  $\Psi_{\text{OUT}}(\lambda, x), \Psi_{\text{IN}}(\lambda, x)$  must match in the region of overlap, provided this is not empty:

$$\Psi_{\text{OUT}}(\lambda, x) \sim \Psi_{\text{IN}}(\lambda, x), \quad |x|^{\delta_{\text{OUT}}} \leq |\lambda| \leq |x|^{\delta_{\text{IN}}}, \quad x \rightarrow 0. \quad (12)$$

This relation is to be intended in the sense that the leading terms of the local behaviour of  $\Psi_{\text{OUT}}$  and  $\Psi_{\text{IN}}$  for  $x \rightarrow 0$  must be equal. This determines a simple relation between the two functions of  $x$  appearing in  $A_0, A_x, A_1, A_0 + A_x$ . (12) also implies that  $\delta_{\text{IN}} \leq \delta_{\text{OUT}}$ .

Practically, to fulfil point (ii), we will match a fundamental solution of (8) for  $\lambda \rightarrow 0$ , with a fundamental solution of (9) when  $\mu := \lambda/x \rightarrow \infty$ , namely with a solution of

$$\frac{d\Psi_{\text{IN}}}{d\mu} = \left[ \frac{A_0}{\mu} + \frac{A_x}{\mu - 1} - x A_1 \sum_{n=0}^{N_{\text{IN}}} x^n \mu^n \right] \Psi_{\text{IN}}, \quad \mu := \frac{\lambda}{x}. \quad (13)$$

To summarize, *matching* two fundamental solutions of the reduced *isomonodromic* systems (8) and (9), we obtain the leading term(s), for  $x \rightarrow 0$ , of the entries of the matrices of the original system (1). The procedure is algorithmic, no *a priori* assumption about the behaviour being necessary.

This method is sometimes called *coalescence of singularities*, because the singularity  $\lambda = 0$  and  $\lambda = x$  coalesce to produce system (8), while the singularity  $\mu = \frac{1}{x}$  and  $\mu = \infty$  coalesce to produce system (13). Coalescence of singularities was first used by Jimbo [16] to compute the monodromy matrices of (1) for a class of solutions of (PVI) with leading term  $y(x) \sim ax^{1-\sigma}, 0 < \text{Re } \sigma < 1$ .

### 2.2. Computation of the monodromy data

Let  $\Psi$  be a fundamental matrix solution of (1), and let  $M_0, M_x, M_1, M_\infty$  be its monodromy matrices at  $\lambda = 0, x, 1, \infty$ , respectively ( $M_\infty$  is the product of  $M_0, M_x, M_1$ , the order depending on the choice of a basis of loops). As a consequence of isomonodromicity, there exists a fundamental solution  $\Psi_{\text{OUT}}$  of (8) such that

$$M_1^{\text{OUT}} = M_1, \quad M_\infty^{\text{OUT}} = M_\infty,$$

where  $M_1^{\text{OUT}}$  and  $M_\infty^{\text{OUT}}$  are the monodromy matrices of  $\Psi_{\text{OUT}}$  at  $\lambda = 1, \infty$ . Moreover,  $M_0^{\text{OUT}} = M_0 M_x$  or  $M_x M_0$ , depending on the order of loops. A detailed proof of these facts can be found in [9]. There also exists a fundamental solution  $\Psi_{\text{IN}}$  of (9) such that

$$M_0^{\text{IN}} = M_0, \quad M_x^{\text{IN}} = M_x,$$

where  $M_0^{\text{IN}}$  and  $M_x^{\text{IN}}$  are the monodromy matrices of  $\Psi_{\text{IN}}$  at  $\lambda = 0, x$ .

The method of coalescence of singularities is useful when the monodromy of the reduced systems (8), (9) can be explicitly computed. This is the case when the reduction is Fuchsian (namely (10), (11)), because Fuchsian systems with three singular points are equivalent to a Gauss hyper-geometric equation (see appendix A). For the non-Fuchsian reduction, in general we can compute the monodromy when (8), (9) are solvable in terms of special or elementary functions. This will be discussed case by case in the paper.

In order for this procedure to work, not only  $\Psi_{\text{OUT}}$  and  $\Psi_{\text{IN}}$  must match with each other, as in subsection 2.1, but also  $\Psi_{\text{OUT}}$  must match with a fundamental matrix solution  $\Psi$  of (1) in a domain of the  $\lambda$ -plane and  $\Psi_{\text{IN}}$  must match with *the same*  $\Psi$  in another domain of the  $\lambda$ -plane.

The standard choice of  $\Psi$  is as follows:

$$\Psi(\lambda) = \begin{cases} \left[ I + O\left(\frac{1}{\lambda}\right) \right] \lambda^{-\frac{\theta_\infty}{2}\sigma_3} \lambda^{R_\infty}, & \lambda \rightarrow \infty; \\ \psi_0(x) [I + O(\lambda)] \lambda^{\frac{\theta_0}{2}\sigma_3} \lambda^{R_0} C_0, & \lambda \rightarrow 0; \\ \psi_x(x) [I + O(\lambda - x)] (\lambda - x)^{\frac{\theta_x}{2}\sigma_3} (\lambda - x)^{R_x} C_x, & \lambda \rightarrow x; \\ \psi_1(x) [I + O(\lambda - 1)] (\lambda - 1)^{\frac{\theta_1}{2}\sigma_3} (\lambda - 1)^{R_1} C_1, & \lambda \rightarrow 1. \end{cases} \quad (14)$$

Here  $\psi_0(x)$ ,  $\psi_x(x)$ ,  $\psi_1(x)$  are the diagonalizing matrices of  $A_0(x)$ ,  $A_1(x)$ ,  $A_x(x)$ , respectively. They are defined by multiplication to the right by arbitrary diagonal matrices, possibly depending on  $x$ .  $C_\kappa$ ,  $\kappa = \infty, 0, x, 1$ , are invertible *connection matrices*, independent of  $x$  [17]. Each  $R_\kappa$ ,  $\kappa = \infty, 0, x, 1$ , is also independent of  $x$ , and

$$R_\kappa = 0 \quad \text{if} \quad \theta_\kappa \notin \mathbf{Z}, \quad R_\kappa = \begin{cases} \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}, & \text{if } \theta_\kappa > 0 \text{ integer;} \\ \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}, & \text{if } \theta_\kappa < 0 \text{ integer.} \end{cases}$$

If  $\theta_i = 0, i = 0, x, 1$ , then  $R_i$  is to be considered the Jordan form  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  of  $A_i$ . If  $\theta_\infty = 0, R_\infty = 0$ . Note that for the loop  $\lambda \mapsto \lambda e^{2\pi i}$ ,  $|\lambda| > \max\{1, |x|\}$ , we immediately compute the monodromy at infinity:

$$M_\infty = \exp\{-i\pi\theta_\infty\} \exp\{2\pi i R_\infty\}.$$

Let  $\Psi_{\text{OUT}}$  and  $\Psi_{\text{IN}}$  be the solutions of (8) and (9) matching as in (12). We explain how they are matched with (14).

(\*) *Matching*  $\Psi \leftrightarrow \Psi_{\text{OUT}}$

$\lambda = \infty$  is a Fuchsian singularity of (8), with residue  $-A_\infty/\lambda$ . Therefore, we can always find a fundamental matrix solution with behaviour

$$\Psi_{\text{OUT}}^{\text{Match}} = \left[ I + O\left(\frac{1}{\lambda}\right) \right] \lambda^{-\frac{\theta_\infty}{2}\sigma_3} \lambda^{R_\infty}, \quad \lambda \rightarrow \infty.$$

This solution matches with  $\Psi$ . Also  $\lambda = 1$  is a Fuchsian singularity of (8). Therefore, we have

$$\Psi_{\text{OUT}}^{\text{Match}} = \psi_1^{\text{OUT}}(x) [I + O(\lambda - 1)] (\lambda - 1)^{\frac{\theta_1}{2}\sigma_3} (\lambda - 1)^{R_1} C_1^{\text{OUT}}, \quad \lambda \rightarrow 1.$$

Here  $C_1^{\text{OUT}}$  is a suitable connection matrix.  $\psi_1^{\text{OUT}}(x)$  is the matrix that diagonalizes the leading terms of  $A_1(x)$ . Therefore,  $\psi_1(x) \sim \psi_1^{\text{OUT}}(x)$  for  $x \rightarrow 0$ . As a consequence of isomonodromicity,  $R_1$  is the same of  $\Psi$ .

As a consequence of the matching  $\Psi \leftrightarrow \Psi_{\text{OUT}}^{\text{Match}}$ , the monodromy of  $\Psi$  at  $\lambda = 1$  is

$$M_1 = C_1^{-1} \exp\{i\pi\theta_1\sigma_3\} \exp\{2\pi i R_1\} C_1 \quad \text{with} \quad C_1 \equiv C_1^{\text{OUT}}.$$

We finally need an invertible connection matrix  $C_{\text{OUT}}$  to connect  $\Psi_{\text{OUT}}^{\text{Match}}$  with the solution  $\Psi_{\text{OUT}}$  appearing in (12). Namely,  $\Psi_{\text{OUT}}^{\text{Match}} = \Psi_{\text{OUT}} C_{\text{OUT}}$ .

(\*) *Matching*  $\Psi \leftrightarrow \Psi_{\text{IN}}$

As a consequence of the matching  $\Psi \leftrightarrow \Psi_{\text{OUT}}^{\text{Match}}$ , we have to choose the IN solution which matches with  $\Psi_{\text{OUT}}^{\text{Match}}$ . This is  $\Psi_{\text{IN}}^{\text{Match}} := \Psi_{\text{IN}} C_{\text{OUT}}$ .

Now,  $\lambda = 0, x$  are Fuchsian singularities of (9). Therefore,

$$\Psi_{\text{IN}}^{\text{Match}} = \begin{cases} \psi_0^{\text{IN}}(x) [I + O(\lambda)] \lambda^{\frac{\theta_0}{2}\sigma_3} \lambda^{R_0} C_0^{\text{IN}}, & \lambda \rightarrow 0; \\ \psi_x^{\text{IN}}(x) [I + O(\lambda - x)] (\lambda - x)^{\frac{\theta_x}{2}\sigma_3} (\lambda - x)^{R_x} C_x^{\text{IN}}, & \lambda \rightarrow x. \end{cases}$$

The above hold for fixed small  $x \neq 0$ . Here,  $C_0^{\text{IN}}$  and  $C_x^{\text{IN}}$  are the suitable connection matrices.  $\psi_0^{\text{IN}}(x)$  and  $\psi_x(x)^{\text{IN}}$  are the diagonalizing matrices of the leading terms of  $A_0(x)$  and  $A_x(x)$ . For  $x \rightarrow 0$  they match with  $\psi_0(x)$  and  $\psi_x(x)$  of  $\Psi$  in (14). On the other hand, as a consequence of isomonodromicity, the matrices  $R_0$  and  $R_x$  are the same as  $\Psi$ .

By virtue of the matching  $\Psi \leftrightarrow \Psi_{\text{IN}}^{\text{Match}}$ , the connection matrices  $C_0$  and  $C_x$  coincide with the  $x$ -independent connection matrices  $C_0^{\text{IN}}, C_x^{\text{IN}}$ , respectively. As a result, we obtain the monodromy matrices for  $\Psi$ :

$$\begin{aligned} M_0 &= C_0^{-1} \exp\{i\pi\theta_0\sigma_3\} \exp\{2\pi iR_0\}C_0, & C_0 &\equiv C_0^{\text{IN}}, \\ M_x &= C_x^{-1} \exp\{i\pi\theta_x\sigma_3\} \exp\{2\pi iR_x\}C_x, & C_x &\equiv C_x^{\text{IN}}. \end{aligned}$$

Our reduction is useful if the connection matrices  $C_1^{\text{OUT}}, C_0^{\text{IN}}, C_x^{\text{IN}}$  can be computed explicitly. This is possible for the Fuchsian-reduced systems (10), (11). For non-Fuchsian-reduced systems, we will discuss the computability case by case.

### 3. Results

In the following, it is understood that  $x \rightarrow 0$  inside a sector. Namely,  $\arg(x)$  is bounded.

#### 3.1. Critical behaviours: result I

The novelty of this paper is that the matching procedure is applied to non-Fuchsian systems (8) and (9). As a result, we obtain all the solutions that admit a Taylor expansion

$$y(x) = b_0 + b_1x + b_2x^2 + \dots = \sum_{n=0}^{\infty} b_nx^n, \quad x \rightarrow 0.$$

Precisely, we obtain the representative solutions of three equivalence classes, the equivalence relation being the birational transformations [23] of appendix C and formula (19). Our result is the following.

**Theorem 1.** *The solutions of (PVI) with Taylor expansion at  $x = 0$  are divided into four equivalent classes (one being that of singular solutions  $y = 0, 1, x$ ). The representatives can be chosen as follows:*

- (1) Singular solution  $y = 1$ .
- (2)  $\theta_\infty \neq 1, \theta_1 - \theta_\infty \notin \mathbf{Z}$  (representative of  $\theta_1 \pm \theta_\infty \notin \mathbf{Z}$ ):

$$\begin{aligned} y(x) &= \frac{\theta_1 - \theta_\infty + 1}{1 - \theta_\infty} + \frac{\theta_1[(\theta_1 - \theta_\infty)(\theta_1 - \theta_\infty + 2) + \theta_x^2 - \theta_0^2]}{2(\theta_\infty - 1)(\theta_\infty - \theta_1)(\theta_\infty - \theta_1 - 2)}x \\ &\quad + \sum_{n=3}^{\infty} b_n(\theta_1, \theta_\infty, \theta_0, \theta_x)x^n. \end{aligned} \tag{15}$$

The coefficients are certain rational functions of  $\theta_0, \theta_\infty, \theta_0, \theta_x$ .

- (3)  $\theta_1 = \theta_\infty \neq 1, \theta_0 = \pm\theta_x$  (representative of  $\theta_1 \pm \theta_\infty \in \mathbf{Z}, \theta_x \pm \theta_0 \in \mathbf{Z}$ ):

$$y(x) = \frac{1}{1 - \theta_\infty} + ax + \sum_{n=2}^{\infty} b_n(a; \theta_0, \theta_\infty)x^n. \tag{16}$$

The coefficients are certain rational functions of  $\theta_0, \theta_\infty$  and a parameter  $a \in \mathbf{C}$ .



(4)  $\theta_\infty = 1, \theta_1 = 0$  (representative of  $\theta_1 \pm \theta_\infty \in \mathbf{Z}, \theta_\infty \in \mathbf{Z} \setminus \{0\}$ ):

$$y(x) = a + \frac{1-a}{2}(1 + \theta_0^2 - \theta_x^2)x + \sum_{n=2}^{\infty} b_n(a; \theta_0; \theta_x)x^n. \tag{17}$$

The coefficients are certain rational functions of  $\theta_0, \theta_x$  and a parameter  $a \in \mathbf{C}$ .

The monodromy data associated with the above solutions are given in theorem 3.

The symmetry  $\theta_1 \mapsto -\theta_1$ , which leaves (PVI) invariant, transforms (15) into

$$y(x) = \frac{\theta_1 + \theta_\infty - 1}{\theta_\infty - 1} + \frac{\theta_1[(\theta_1 + \theta_\infty)(\theta_1 + \theta_\infty - 2) + \theta_x^2 - \theta_0^2]}{2(1 - \theta_\infty)(\theta_\infty + \theta_1)(\theta_\infty + \theta_1 - 2)}x + \sum_{n=3}^{\infty} b_n(-\theta_1, \theta_\infty, \theta_0, \theta_x)x^n. \tag{18}$$

Here  $\theta_\infty \neq 1, \theta_1 + \theta_\infty \notin \mathbf{Z}$ . The coefficients  $b_n$  are the same as (15).

The convergence of the Taylor series can be proved by a Briot–Bouquet-like argument. This will not be done here, for reasons of space. The reader can find the general procedure in [14] and an application to the fifth Painlevé equation in [19]

*Comments*

(1) *Characterization of solutions*  $y(x) = \sum_{n=0}^{\infty} b_n x^n, b_0 \neq 0$ .

(a) There always exists one solution (15) when  $\theta_1 - \theta_\infty \notin \mathbf{Z}$ ; there always exists one solution (18) when  $\theta_1 + \theta_\infty \notin \mathbf{Z}$ . The coefficients  $b_n$  depend rationally on  $\theta_\kappa, \kappa = 0, x, 1, \infty$ .

(b) There is a one-parameter family of solutions equivalent to (16), when  $\theta_1 \pm \theta_\infty \in \mathbf{Z}$  and  $\theta_0 \pm \theta_x$  has a particular integer value. The coefficients  $b_n$  depend rationally on a complex parameter  $a$  and  $\theta_\infty, \theta_0$ . (c) Finally, there is a one-parameter family of solutions equivalent to (17), when  $\theta_1 \pm \theta_\infty \in \mathbf{Z}$ , and  $\theta_\infty$  has a particular integer value; the coefficients  $b_n$  depend rationally on a complex parameter  $a$  and  $\theta_0, \theta_x$ . The singular solutions  $y = 0, 1, x$  are possibly obtained by birational transformations of (15), (16), (17).

The coefficients  $b_n$  can always be computed recursively by direct substitution into (PVI). We will clarify these facts by some examples in appendix D.

(2) *Characterization of solutions*  $y(x) = \sum_{n=1}^{\infty} b_n x^n, b_1 \neq 0$ .

These solutions are obtained from those of theorem 1 by the symmetry:

$$\begin{aligned} \theta_x &\mapsto \theta_1, & \theta_0 &\mapsto \theta_\infty - 1, & \theta_1 &\mapsto \theta_x, \\ \theta_\infty &\mapsto \theta_0 + 1; & y(x) &\mapsto \frac{x}{y(x)}. \end{aligned} \tag{19}$$

The solutions obtained from the singular solution  $y = 1$  and (15), (16), (17) are, respectively, as follows:

(1) Singular solution  $y(x) = x$ .

(2)  $\theta_0 \neq 0, \theta_0 \pm \theta_x \notin \mathbf{Z}$ :

$$y(x) = \frac{\theta_0}{\theta_0 \pm \theta_x}x \pm \frac{\theta_0\theta_x [(\theta_0 \pm \theta_x)^2 + \theta_1^2 - \theta_\infty^2 + 2\theta_\infty - 2]}{2(\theta_0 \pm \theta_x)^2[(\theta_0 \pm \theta_x)^2 - 1]}x^2 + \sum_{n=3}^{\infty} b_n(\theta_0, \theta_x, \theta_1, \theta_\infty)x^n. \tag{20}$$

(3)  $\theta_0 + \theta_x = 1, \theta_0 \neq 0, \theta_1 = \pm(\theta_\infty - 1)$ :

$$y(x) = \theta_0x + ax^2 + \sum_{n=3}^{\infty} b_n(a; \theta_0, \theta_\infty)x^n. \tag{21}$$

(4)  $\theta_x = \theta_0 = 0$ :

$$y(x) = ax + \frac{a(a-1)}{2}(\theta_1^2 - (\theta_\infty - 1)^2 - 1)x^2 + \sum_{n=3}^{\infty} b_n(a; \theta_1, \theta_\infty)x^n. \tag{22}$$

(a) (PVI) has always one or both solutions (20) when  $\theta_0 \pm \theta_x \notin \mathbf{Z}$ . Also when  $\theta_0 + \theta_x$  (or  $\theta_0 - \theta_x$ ) is integer, (PVI) has a solution (20) corresponding to  $\theta_0 - \theta_x$  not integer (or  $\theta_0 + \theta_x$  not integer). (b) When  $\theta_0 + \theta_x$  or  $\theta_0 - \theta_x$  is integer, (PVI) has a one-parameter family of solutions equivalent (by birational transformations) to (21); this family exists provided that  $\theta_1 \pm \theta_\infty$  has a particular integer value. (c) When  $\theta_0 + \theta_x$  or  $\theta_0 - \theta_x$  is integer and  $\theta_0$  has a particular integer value, there is a one-parameter family of solutions equivalent to (22).

(3) (PVI) has a one-parameter family of solutions of the type

$$y(x) = y_0(x) + y_1(x)ax^\omega + y_2(x)(ax^\omega)^2 + \dots = \sum_{N=0}^{\infty} y_N(x)(ax^\omega)^N, \quad x \rightarrow 0, \tag{23}$$

where the parameter is  $a \in \mathbf{C}$ , and  $y_N(x)$ 's are Taylor series:

$$y_N(x) = \sum_{k=0}^{\infty} b_{k,N}(\theta_1, \theta_\infty, \theta_0, \theta_x)x^k, \quad x \rightarrow 0.$$

Either  $y_0(x)$  is (18) and  $\omega = \pm(\theta_1 + \theta_\infty - 1)$  or  $y_0(x)$  is (15) and  $\omega = \pm(\theta_\infty - \theta_1 - 1)$ . The conditions  $|\operatorname{Re} \omega| < 1, \omega \neq 0$  hold. The coefficients  $b_{k,N}(\theta_1, \theta_\infty, \theta_0, \theta_x)$  are certain rational functions that can be recursively determined by direct substitution into (PVI). These solutions are the images of solutions (25) and (26), respectively, through the symmetry (19). Solutions (25) and (26) are a sub-case of theorem 2, obtained by the matching procedure.

Taylor solutions (15), (18) are a special case of (23), when the parameter is zero. Solutions (16) and (17)—and their images by symmetry—are one-parameter families of Type (23), in non-generic cases when  $\omega \in \mathbf{Z}$ .

Further study of one-parameter solutions, including non-generic cases when  $\theta_v$  and/or some sum of two  $\theta_v$ 's are integer (including logarithmic one-parameter families), will be presented in another paper devoted to the general classification problem.

(4) Solutions (15) and the equivalent solutions (18), (20) were also derived in [18] by substitution of a Taylor expansion in (PVI). The corresponding monodromy was computed by coalescence of singularities of a Heun's type (scalar) equation.

### 3.2. Critical behaviours: result II

We now consider cases when (1) can be reduced to the Fuchsian systems (10) and (11). Let  $\sigma$  be a complex number defined, up to sign, by

$$\operatorname{tr}(M_0 M_x) = 2 \cos(\pi \sigma), \quad |\operatorname{Re} \sigma| \leq 1.$$

Actually,  $\pm\sigma/2$  are the eigenvalues of  $\lim_{x \rightarrow 0}(A_0 + A_x)$ . The matching procedure yields the following result.

**Theorem 2.** *Let  $r \in \mathbf{C}$  and  $\sigma$  be as above, with the restriction  $|\operatorname{Re} \sigma| < 1$ . (PVI) has a family of solutions depending on the two parameters  $r, \sigma$ . The leading terms of the critical behaviour for  $x \rightarrow 0$  may be parameterized as follows:*

For  $\sigma \neq 0$ ,

$$y(x) \sim \begin{cases} \frac{1}{r} \frac{[\sigma^2 - (\theta_0 + \theta_x)^2][(\theta_0 - \theta_x)^2 - \sigma^2]}{16\sigma^3} x^{1-\sigma}, & \text{if } \operatorname{Re} \sigma > 0; \\ -\frac{r}{\sigma} x^{1+\sigma}, & \text{if } \operatorname{Re} \sigma < 0; \\ x \left\{ iA \sin(i\sigma \ln x + \phi) + \frac{\theta_0^2 - \theta_x^2 + \sigma^2}{2\sigma^2} \right\}, & \text{if } \operatorname{Re} \sigma = 0. \end{cases} \quad (24)$$

In the above formulae,  $r \neq 0$  and

$$\phi := i \ln \frac{2r}{\sigma A}, \quad A := \left[ \frac{\theta_0^2}{\sigma^2} - \left( \frac{\theta_0^2 - \theta_x^2 + \sigma^2}{2\sigma^2} \right)^2 \right]^{\frac{1}{2}}.$$

For special values of  $\sigma \neq 0$ ,

$$y(x) \sim \frac{\theta_0}{\theta_0 + \theta_x} x \mp \frac{r}{\theta_0 + \theta_x} x^{1+\sigma}, \quad \sigma = \pm(\theta_0 + \theta_x) \neq 0, \quad (25)$$

$$y(x) \sim \frac{\theta_0}{\theta_0 - \theta_x} x \mp \frac{r}{\theta_0 - \theta_x} x^{1+\sigma}, \quad \sigma = \pm(\theta_0 - \theta_x) \neq 0. \quad (26)$$

For  $\sigma = 0$ ,

$$y(x) \sim \begin{cases} x \left\{ \frac{\theta_x^2 - \theta_0^2}{4} \left[ \ln x + \frac{4r + 2\theta_0}{\theta_0^2 - \theta_x^2} \right]^2 + \frac{\theta_0^2}{\theta_0^2 - \theta_x^2} \right\}, & \theta_0 \neq \pm\theta_x, \\ x(r \pm \theta_0 \ln x), & \theta_0 = \pm\theta_x. \end{cases} \quad (27)$$

### Comments

- (1)  $r$  can be computed as a function of the monodromy data. See (34) and comments there. The sign (branch) of the two square roots appearing in  $\phi$  and  $\mathbf{a}$  is the same.  $x \rightarrow 0$  in a sector of width less than  $2\pi$ .
- (2) *Sub-cases of theorem 2*

- (i) When  $\sigma \neq 0$ , the result of the theorem includes sub-cases (25) and (26). If  $r = 0$ ,  $\theta_0 \neq 0, \theta_0 \pm \theta_x \notin \mathbf{Z}$ , direct substitution into (PVI) gives the two Taylor expansions (20).

If  $r \neq 0$ , (25) and (26) are a one-parameter family, with the restriction  $|\operatorname{Re} \sigma| < 1$ . The symmetry (19) transforms them into solutions (23), the leading terms being, respectively,

$$y(x) \sim \frac{\theta_\infty + \theta_1 - 1}{\theta_\infty - 1} \left( 1 \pm \frac{r}{\theta_\infty - 1} x^\omega \right), \quad \omega = \pm(\theta_\infty + \theta_1 - 1) \neq 0,$$

$$y(x) \sim \frac{\theta_\infty - \theta_1 - 1}{\theta_\infty - 1} \left( 1 \pm \frac{r}{\theta_\infty - 1} x^\omega \right), \quad \omega = \pm(\theta_\infty - \theta_1 - 1) \neq 0,$$

with the restriction  $|\operatorname{Re} \omega| < 1$ .

- (ii) The case  $\sigma = 0$  includes the sub-case  $y(x) \sim rx$ , which occurs for  $\theta_0 = \theta_x, \theta_0 = 0$ . By direct substitution in (PVI) we obtain a series

$$y(x) = rx + \sum_{n=3}^{\infty} b_n(r, \theta_1, \theta_\infty) x^n, \quad \theta_0 = \theta_x = 0, \quad r \neq 0, 1.$$

This is again solution (22). Note that for  $r = 0, 1$  we have the singular solutions  $y = 0, y = 1$ . Also note that the special sub-sub-case  $\theta_0 = \theta_x = \theta_1 = 0$  has applications in the theory of *semi-simple Frobenius manifolds* of dimension 3 [7, 10].

- (3) Solutions (24) were studied in [16]. Their existence was proved by assuming that the matrices  $A_0, A_x, A_1$  have a certain critical behaviour for  $x \rightarrow 0$  and proving that such matrices solve the Schlesinger equations. Then, the monodromy data were computed by coalescence of singularity. These solutions were further studied in [5, 8, 9, 11]. We show that these solutions can be obtained without any assumption by the matching procedure, together with solutions (27), which do not appear in [16].
- (4) The class of solutions (24) was enlarged in [11, 25], to the values  $\sigma \in \mathbf{C}, \sigma \notin (-\infty, 0] \cup [1, +\infty)$ . When  $\text{Re } \sigma \geq 1$  or  $\text{Re } \sigma \leq 0$ , the critical behaviour is like the first of (24) and it holds for  $x \rightarrow 0$  in a spiral-shaped domain in the universal covering of a punctured neighbourhood of  $x = 0$ , along a paths joining a point  $x_0$  to  $x = 0$ . Along special paths which approach the movable poles, these solutions may have behaviour  $y(x) \sim \sin^{-2}\left(\frac{i\sigma}{2} \ln x + \varphi(x, r)\right)$ , where  $\varphi(x, r)$  is a phase depending on the parameter  $r$ . The transformation  $\sigma \mapsto \pm\sigma + 2N, N \in \mathbf{Z}$ , leaves the identity  $\text{tr}(M_0 M_x) = 2 \cos(\pi\sigma)$  invariant. Its effect on the solutions is studied in [11]. As a result, one can reduce to the values  $0 \leq \text{Re } \sigma \leq 1, \sigma \neq 0, 1$ . We cannot enter into more details here. The reader may find a synthetic description of these results in the review paper [12].
- (5) Solutions with expansion

$$y(x) = x(A_1 + B_1 \ln x + C_1 \ln^2 x + D_1 \ln^3 x + \dots) + x^2(A_2 + B_2 \ln x + \dots) + \dots, \quad x \rightarrow 0$$

are all included in theorems 1 and 2. Actually, only the following cases are possible:

$$y(x) = \begin{cases} \frac{\theta_0}{\theta_0 \pm \theta_x} x + O(x^2) & \text{(Taylor expansion),} \\ x \left( \frac{\theta_0^2 - B_1^2}{\theta_0^2 - \theta_x^2} + B_1 \ln x + \frac{\theta_x^2 - \theta_0^2}{4} \ln^2 x \right) + x^2(\dots) + \dots, & (28) \\ x(A_1 \pm \theta_0 \ln x) + x^2(\dots) + \dots & \text{and } \theta_0 = \pm \theta_x. \end{cases}$$

$A_1$  and  $B_1$  are the parameters. We see that the higher orders in (27) are  $O(x^2 \ln^m x)$ , for some integer  $m > 0$ .

- (6) The symmetry (19) applied to solutions (27) gives

$$y(x) \sim 4 \left( (\theta_1^2 - (\theta_\infty - 1)^2) \left[ \ln x - \frac{4r + 2(\theta_\infty - 1)}{\theta_1^2 - (\theta_\infty - 1)^2} \right]^2 - \frac{4(\theta_\infty - 1)^2}{\theta_1^2 - (\theta_\infty - 1)^2} \right)^{-1},$$

namely,

$$y(x) = \frac{4}{[\theta_1^2 - (\theta_\infty - 1)^2] \ln^2 x} \left[ 1 + \frac{8r + 4(\theta_\infty - 1)}{\theta_1^2 - (\theta_\infty - 1)^2} \frac{1}{\ln x} + O\left(\frac{1}{\ln^2 x}\right) \right], \quad (29)$$

and

$$y(x) = \frac{\pm 1}{(\theta_\infty - 1) \ln x} \left[ 1 \mp \frac{r}{(\theta_\infty - 1) \ln x} + O\left(\frac{1}{\ln^2 x}\right) \right], \quad \theta_\infty \mp \theta_1 = 1.$$

The higher orders  $O(1/\ln^2 x)$  include powers  $x^n (\ln x)^{\pm m}$ . The so-called *Chazy solutions*, studied in [20] for the special case  $\theta_0 = \theta_x = \theta_1 = 0, \theta_\infty = -1$ , have the behaviour (29).

- (7) When this paper was completed, I received a communication by the first author of [6]. In [6] it is proved that (PVI) has solutions with expansion at  $x = \infty$ , or  $x = 0$ , of the form  $y = c_r x^r + \sum_s c_s x^s, c_r \in \mathbf{C}$ .  $c_s$ 's are either complex constants or polynomials in  $\ln x$ .  $r$  and  $s$  are integer or complex. If  $r$  is complex, the restriction  $\text{Re } r \in (0, 1)$  holds. The method used in [6] is a power geometry technique. The connection problem and the characterization of the associated monodromy data are not studied.

(8) When this paper was already accepted for publication, I received a communication about a recent work [24] on the asymptotics of the *real* solutions of (PVI). The asymptotic behaviours obtained in [24] are of the type of our theorem 2, namely (24) and the first behaviour in (27). The tool used is a method of successive approximations. So, the results are local, and the connection problem is not studied. Moreover, some genericity conditions on the coefficients of PVI seem to be necessary (so, for example, the second solution in (27) cannot be obtained).

3.3. Monodromy: result III

In this paper, we computed the monodromy for the Taylor-expanded solutions, which correspond to non-Fuchsian reductions of system (1). Because of the symmetries of (PVI), we can limit ourselves to the monodromy data for the representative solutions (15)–(17).

**Theorem 3.** (a) Let  $\theta_\kappa \notin \mathbf{Z}$ ,  $\kappa = 0, 1, x, \infty$ . A representation for the monodromy matrices of solution (15) is

$$\begin{aligned} M_0 &= C_{0\infty} \exp\{i\pi\theta_0\sigma_3\}C_{0\infty}^{-1}, & M_x &= C_{0\infty}C_{01}^{-1} \exp\{i\pi\theta_x\sigma_3\}C_{01}C_{0\infty}^{-1}, \\ M_1 &= \exp\{-i\pi\theta_1\sigma_3\}, & M_\infty &= \exp\{-i\pi\theta_\infty\sigma_3\}. \end{aligned}$$

The matrices  $C_{0\infty}$  and  $C_{01}$  are

$$C_{0\infty} := \begin{bmatrix} \frac{\Gamma\left(1+\frac{\theta_1-\theta_\infty}{2}\right)\Gamma(1+\theta_0) e^{i\frac{\pi}{2}[\theta_0+\theta_x+\theta_\infty-\theta_1]}}{\Gamma\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{\theta_1-\theta_\infty}{2}+1\right)\Gamma\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}+\frac{\theta_1-\theta_\infty}{2}+1\right)} & \frac{\Gamma\left(1+\frac{\theta_1-\theta_\infty}{2}\right)\Gamma(1-\theta_0) e^{i\frac{\pi}{2}[\theta_x-\theta_0+\theta_\infty-\theta_1]}}{\Gamma\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}-\frac{\theta_\infty}{2}+\frac{\theta_1}{2}+1\right)\Gamma\left(\frac{\theta_x}{2}-\frac{\theta_0}{2}+\frac{\theta_1-\theta_\infty}{2}+1\right)} \\ -\frac{\Gamma\left(\frac{\theta_\infty}{2}-\frac{\theta_1}{2}-1\right)\Gamma(1+\theta_0) e^{i\frac{\pi}{2}[\theta_0+\theta_x+\theta_1-\theta_\infty]}}{\Gamma\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{\theta_\infty}{2}-\frac{\theta_1}{2}\right)\Gamma\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}+\frac{\theta_\infty}{2}-\frac{\theta_1}{2}\right)} & -\frac{\Gamma\left(\frac{\theta_\infty}{2}-\frac{\theta_1}{2}-1\right)\Gamma(1-\theta_0) e^{i\frac{\pi}{2}[\theta_x-\theta_0+\theta_1-\theta_\infty]}}{\Gamma\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}-\frac{\theta_1}{2}+\frac{\theta_\infty}{2}\right)\Gamma\left(\frac{\theta_x}{2}-\frac{\theta_0}{2}+\frac{\theta_\infty}{2}-\frac{\theta_1}{2}\right)} \end{bmatrix}, \quad (30)$$

$$C_{01} := \begin{bmatrix} \frac{\Gamma(-\theta_x)\Gamma(1+\theta_0)}{\Gamma\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}+\frac{\theta_1-\theta_\infty}{2}+1\right)\Gamma\left(\frac{\theta_0}{2}-\frac{\theta_x}{2}+\frac{\theta_\infty}{2}-\frac{\theta_1}{2}\right)} & \frac{\Gamma(-\theta_x)\Gamma(1-\theta_0)}{\Gamma\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}-\frac{\theta_\infty}{2}+\frac{\theta_1}{2}+1\right)\Gamma\left(-\frac{\theta_0}{2}-\frac{\theta_x}{2}-\frac{\theta_1}{2}+\frac{\theta_\infty}{2}\right)} \\ \frac{\Gamma(\theta_x)\Gamma(1+\theta_0)}{\Gamma\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{\theta_\infty}{2}-\frac{\theta_1}{2}\right)\Gamma\left(\frac{\theta_0}{2}+\frac{\theta_x}{2}+\frac{\theta_1-\theta_\infty}{2}+1\right)} & \frac{\Gamma(\theta_x)\Gamma(1-\theta_0)}{\Gamma\left(\frac{\theta_x}{2}-\frac{\theta_0}{2}+\frac{\theta_\infty}{2}-\frac{\theta_1}{2}\right)\Gamma\left(\frac{\theta_x}{2}-\frac{\theta_0}{2}+\frac{\theta_1-\theta_\infty}{2}+1\right)} \end{bmatrix}. \quad (31)$$

The subgroup generated by  $M_0M_x$  and  $M_1$  is reducible. As for solution (18), we just need to change  $\theta_1 \mapsto -\theta_1$ .

(b) It is convenient to re-parameterize solution (16) by introducing a parameter  $s$  through the equality

$$a = \frac{\theta_\infty(2s + \theta_x + 1)}{2(\theta_\infty - 1)}.$$

Let  $\theta_x, \theta_\infty \notin \mathbf{Z}$ . Then, a representation for the monodromy group is

$$\begin{aligned} M_0 &= G \exp\{i\pi\theta_x\sigma_3\}G^{-1}, & M_1 &= \exp\{-i\pi\theta_\infty\sigma_3\} \\ M_x &= G \exp\{-i\pi\theta_x\sigma_3\}G^{-1}, & M_\infty &= \exp\{-i\pi\theta_\infty\sigma_3\}. \end{aligned}$$

In particular,  $M_1 = M_\infty, M_0M_x = I$ . We can choose  $G$  as follows:

$$G = \begin{pmatrix} 1 & 1 \\ \frac{s+\theta_x}{r} & \frac{s}{r} \end{pmatrix}.$$

Conversely, we may express  $s$  as a function of the monodromy data:

$$s = \frac{\theta_x[2 \cos(\pi(\theta_\infty + \theta_x)) - \text{tr}(M_1M_0)]}{2[\cos(\pi(\theta_\infty - \theta_x)) - \cos(\pi(\theta_\infty + \theta_x))]}.$$

(c) We re-parameterize solution (17) introducing a new parameter  $s$  defined by  $a =: (1 - s)^{-1}$ . Let  $\theta_0, \theta_x \notin \mathbf{Z}$ . Then, a monodromy representation for solutions (17) is

$$M_0 = (C_{\infty 0})^{-1} \exp\{i\pi\theta_0\sigma_3\}C_{\infty 0}, \quad M_\infty = \begin{pmatrix} -1 & 0 \\ 2\pi i(1 - s) & -1 \end{pmatrix}$$

$$M_x = (C_{\infty 0})^{-1}(C_{01})^{-1} \exp\{i\pi\theta_x\sigma_3\}C_{01}C_{\infty 0}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ 2\pi i s & 1 \end{pmatrix},$$

where  $C_{\infty 0}$  and  $C_{01}$  are (32) and (33). Conversely, we may express  $s$  as a function of the monodromy data:

$$s = \frac{\text{tr}(M_1 M_0) - 2 \cos(\pi\theta_0) (C_{\infty 0})_{21}}{4\pi \sin(\pi\theta_0) (C_{\infty 0})_{22}}.$$

The matrices  $C_{\infty 0}$  and  $C_{01}$  are

$$C_{\infty 0} = 2 \begin{pmatrix} 0 & \frac{\Gamma(-\theta_0) e^{-i\pi(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2})}}{\Gamma(-\frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{3}{2}) \Gamma(-\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2})} \\ \frac{\Gamma(-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{1}{2}) \Gamma(-\frac{\theta_0}{2} + \frac{\theta_x}{2} - \frac{1}{2})}{\Gamma(1-\theta_0) e^{-i\pi(\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{3}{2})}} & \frac{\Gamma(\theta_0) e^{-i\pi(-\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2})}}{\Gamma(\frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{3}{2}) \Gamma(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2})} \end{pmatrix}, \quad (32)$$

$$C_{01} = \begin{pmatrix} \frac{\Gamma(-\theta_x) \Gamma(1+\theta_0)}{\Gamma(\frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{3}{2}) \Gamma(\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{1}{2})} & \frac{\Gamma(-\theta_x) \Gamma(1-\theta_0)}{\Gamma(-\frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{3}{2}) \Gamma(-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{1}{2})} \\ \frac{\Gamma(\theta_x) \Gamma(1+\theta_0)}{\Gamma(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}) \Gamma(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{1}{2})} & \frac{\Gamma(\theta_x) \Gamma(1-\theta_0)}{\Gamma(-\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}) \Gamma(-\frac{\theta_0}{2} + \frac{\theta_x}{2} - \frac{1}{2})} \end{pmatrix}. \quad (33)$$

The conditions  $\theta_\kappa \notin \mathbf{Z}$  can be eliminated, and the computations can be repeated without conceptual changes, but with different results.

In the above theorem, the subgroups generated by  $M_0 M_x$  and  $M_1$  are reducible. This characterizes the monodromy associated with solutions which have a Taylor series at  $x = 0$ . The same characterization at  $x = 1$  involves the subgroup generated by  $M_1 M_x$  and  $M_0$ . At  $x = \infty$ , it involves the subgroup generated by  $M_0 M_1$  and  $M_x$ .<sup>2</sup> In another paper, we will again consider this characterization, together with the general problem of classification.

Let us define again  $\sigma$  by  $\text{tr}(M_0 M_x) = 2 \cos \pi \sigma$ . Then, in case (a),  $\sigma = \pm(\theta_1 - \theta_\infty)$  (and  $\pm(\theta_1 + \theta_\infty)$  for the change  $\theta_1 \mapsto -\theta_1$ ). In case (b),  $\text{tr}(M_0 M_x) = 2$  and  $\sigma = 0$ . In case (c),  $\text{tr}(M_0 M_x) = -2$ ,  $\sigma = \pm 1$ . The matching procedure is effective to produce solutions corresponding to monodromy data for which the connection problem is *so far* not well studied, such as the case  $\text{tr}(M_i M_j) = -2$  (see [12]<sup>3</sup>).

Note. Also the one-parameter solutions (25), (26) and the second solution in (27) are characterized by a reducible subgroup generated by  $M_0, M_x$ .

Comments

- (1) The monodromy group for solutions (20) was derived also in [18], by the confluence of singularities of scalar equations (including a Heun’s type equation). The result is equivalent to that in point (a) of the above theorem.

<sup>2</sup> In the appendix of [11], the reader may find explanations about how to obtain results at  $x = 1, \infty$  from the results at  $x = 0$ .

<sup>3</sup> Here, I remark that formula (1.30), p 1293, of my paper [11] is wrong. The correct one is  $\text{tr}(M_i M_j) \notin (-\infty, -2]$ . In [11], the connection problem is solved for  $\text{tr}(M_i M_j) \neq \pm 2$ . The case  $\text{tr}(M_i M_j) = 2$  yields (27). For the special choice of the parameters  $\theta_0 = \theta_x = \theta_1 = 0$ , it was studied in [8, 9] (no logarithmic terms appear in such a special case). The result (27) for the general (PVI), corresponding to  $\text{tr}(M_0 M_x) = 2$ , appears in the present paper for the first time.

(2) The computation of the monodromy group of the Fuchsian systems (10) and (11) is quite clear [5, 8, 11, 16]. It allows us to express the parameter  $r$  of (24)–(27) as a function of the monodromy data. Though the computation for (25)–(27) does not appear in the literature, the procedure is clear (see section 4.8), so we do not repeat it. We just report the result for (24), which can be found in [5, 11, 16]:

$$r = \frac{(\theta_0 - \theta_x + \sigma)(\theta_0 + \theta_x - \sigma)(\theta_\infty + \theta_1 - \sigma)}{4\sigma(\theta_\infty + \theta_1 + \sigma)} \frac{1}{\mathbf{F}}, \tag{34}$$

where

$$\mathbf{F} := \frac{\Gamma(1 + \sigma)^2 \Gamma\left(\frac{1}{2}(\theta_0 + \theta_x - \sigma) + 1\right) \Gamma\left(\frac{1}{2}(\theta_x - \theta_0 - \sigma) + 1\right)}{\Gamma(1 - \sigma)^2 \Gamma\left(\frac{1}{2}(\theta_0 + \theta_x + \sigma) + 1\right) \Gamma\left(\frac{1}{2}(\theta_x - \theta_0 + \sigma) + 1\right)} \\ \times \frac{\Gamma\left(\frac{1}{2}(\theta_\infty + \theta_1 - \sigma) + 1\right) \Gamma\left(\frac{1}{2}(\theta_1 - \theta_\infty - \sigma) + 1\right) V}{\Gamma\left(\frac{1}{2}(\theta_\infty + \theta_1 + \sigma) + 1\right) \Gamma\left(\frac{1}{2}(\theta_1 - \theta_\infty + \sigma) + 1\right) U}$$

and

$$U := \left[ \frac{i}{2} \sin(\pi\sigma) \operatorname{tr}(M_1 M_x) - \cos(\pi\theta_x) \cos(\pi\theta_\infty) - \cos(\pi\theta_0) \cos(\pi\theta_1) \right] e^{i\pi\sigma} \\ + \frac{i}{2} \sin(\pi\sigma) \operatorname{tr}(M_0 M_1) + \cos(\pi\theta_x) \cos(\pi\theta_1) + \cos(\pi\theta_\infty) \cos(\pi\theta_0) \\ V := 4 \sin \frac{\pi}{2}(\theta_0 + \theta_x - \sigma) \sin \frac{\pi}{2}(\theta_0 - \theta_x + \sigma) \sin \frac{\pi}{2}(\theta_\infty + \theta_1 - \sigma) \sin \frac{\pi}{2}(\theta_\infty - \theta_1 + \sigma).$$

The above formula was computed with the assumption that  $\sigma \pm (\theta_0 + \theta_x)$ ,  $\sigma \pm (\theta_0 - \theta_x)$ ,  $\sigma \pm (\theta_1 + \theta_\infty)$ ,  $\sigma \pm (\theta_1 - \theta_\infty)$  are not even integers<sup>4</sup>.

(3) *Reducible monodromy.* The monodromy groups in theorem 3 are not reducible, but they have a reducible subgroup. If the entire group itself is completely reducible, all the Painlevé transcendents are known. Solutions of (PVI) corresponding to a reducible monodromy were found in [13]. We summarize the results:

**Proposition 1.** *All the solutions of (PVI) corresponding to a reducible monodromy group are equivalent by birational canonical transformations to the following one-parameter family of solutions, with  $\theta_\infty + \theta_1 + \theta_0 + \theta_x = 0$ :*

$$y(x) = \frac{\theta_1 + \theta_\infty - 1 + x(1 + \theta_x)}{\theta_\infty - 1} - \frac{1}{\theta_\infty - 1} \frac{x(1 - x)}{u(x; a)} \frac{du(x; a)}{dx}, \tag{35}$$

where  $u(x; a) = u_1(x) + au_2(x)$ ;  $a \in \mathbf{C}$ ,  $u_1(x)$  and  $u_2(x)$  are linear independent solutions of the hyper-geometric equation:

$$x(1 - x) \frac{d^2u}{dx^2} + \{[2 - (\theta_\infty + \theta_1)] - (4 - \theta_\infty + \theta_x)x\} \frac{du}{dx} - (2 - \theta_\infty)(1 + \theta_x)u = 0.$$

The monodromy matrices are

$$M_0 = \begin{pmatrix} \frac{\theta_0}{2} & * \\ 0 & -\frac{\theta_0}{2} \end{pmatrix}, \quad M_x = \begin{pmatrix} \frac{\theta_x}{2} & * \\ 0 & -\frac{\theta_x}{2} \end{pmatrix}, \quad M_1 = \begin{pmatrix} \frac{\theta_1}{2} & * \\ 0 & -\frac{\theta_1}{2} \end{pmatrix}.$$

The parameter  $a$  does not appear in the monodromy.

<sup>4</sup> In [11], there is a misprint in formula (A.30), which must be re-calculated. In [16], in formula (1.8) at the bottom of p 1141, the last sign is  $\pm\sigma$  instead of  $\mp\sigma$ .

**Remark.** The *rational solutions* of (PVI) are a special case of the above proposition. They were studied in [21]. Up to canonical birational transformations, they are realized for  $\theta_\infty + \theta_1 + \theta_0 + \theta_x = 0$  and

$$\begin{aligned} \theta_0 = 1: \quad & y(x) = \frac{\theta_\infty + \theta_1}{\theta_\infty} \frac{x - 1}{x(1 + \theta_1) - (\theta_1 + \theta_\infty)}; \\ \theta_0 = -2: \quad & y(x) = \frac{(2 - (\theta_\infty + \theta_1) + \theta_1 x)^2 - 2 + \theta_\infty + \theta_1 - \theta_1 x^2}{(1 - \theta_\infty)(2 - (\theta_\infty + \theta_1) + \theta_1 x)}. \end{aligned}$$

The computation of the expansion at  $x = 0$  of (35) is just a consequence of the expansions of  $u_1(x)$  and  $u_2(x)$ . The reader can find by himself a behaviour  $y \sim x(r(a) \pm \theta_x \ln(x))$  for  $\theta_1 + \theta_\infty = \theta_0 + \theta_x = 0$ , namely, a sub-case of the second solution in (27). For  $\theta_1 + \theta_\infty \notin \mathbf{Z}$ , we find behaviours of the type (23) (and (15), (20) for  $a = 0$ ).

### Part II. Derivation results: Fuchsian reduction

#### 4. Fuchsian case

Let  $x \rightarrow 0$ . The reduction to the Fuchsian systems (10) is possible if in the domain (4) we have

$$|(A_0 + A_x)_{ij}| \gg \left| (A_x)_{ij} \frac{x}{\lambda} \right|, \quad \text{namely, } |(A_0 + A_x)_{ij}| \gg |(A_x)_{ij} x^{1-\delta_{\text{OUT}}}|. \tag{36}$$

Let us denote with  $\hat{A}_i$  the leading term of the matrix  $A_i$ ,  $i = 0, x, 1$ . We can substitute (10) with

$$\frac{d\Psi_{\text{OUT}}}{d\lambda} = \left[ \frac{\hat{A}_0 + \hat{A}_x}{\lambda} + \frac{\hat{A}_1}{\lambda - 1} \right] \Psi_{\text{OUT}}. \tag{37}$$

We suppose that  $\theta_\infty \neq 0$ . This is not a loss in generality, because  $\theta_\infty = 0$  is equivalent to  $\theta_\infty = 2$ .

**Lemma 1.** *If the approximation (10) is possible, then  $\hat{A}_0 + \hat{A}_x$  has eigenvalues  $\pm \frac{\sigma}{2} \in \mathbf{C}$  independent of  $x$ , defined (up to sign and addition of an integer) by  $\text{tr}(M_x M_0) = 2 \cos(\pi \sigma)$ . Let  $r_1 \in \mathbf{C}, r_1 \neq 0$ . For  $\theta_\infty \neq 0$ , the leading terms are*

$$\hat{A}_1 = \begin{pmatrix} \frac{\sigma^2 - \theta_\infty^2 - \theta_1^2}{4\theta_\infty} & -r_1 \\ \frac{[\sigma^2 - (\theta_1 - \theta_\infty)^2][\sigma^2 - (\theta_1 + \theta_\infty)^2]}{16\theta_\infty^2} \frac{1}{r_1} & -\frac{\sigma^2 - \theta_\infty^2 - \theta_1^2}{4\theta_\infty} \end{pmatrix}, \tag{38}$$

and

$$\hat{A}_0 + \hat{A}_x = \begin{pmatrix} \frac{\theta_1^2 - \sigma^2 - \theta_\infty^2}{4\theta_\infty} & r_1 \\ -\frac{[\sigma^2 - (\theta_1 - \theta_\infty)^2][\sigma^2 - (\theta_1 + \theta_\infty)^2]}{16\theta_\infty^2} \frac{1}{r_1} & -\frac{\theta_1^2 - \sigma^2 - \theta_\infty^2}{4\theta_\infty} \end{pmatrix}. \tag{39}$$

**Proof.** Observe that  $\text{tr}(\hat{A}_0 + \hat{A}_x) = \text{tr}(A_0 + A_x) = 0$ , thus, for any  $x$ ,  $\hat{A}_0 + \hat{A}_x$  has eigenvalues of opposite sign, that we denote as  $\pm \tilde{\sigma}(x)/2$ . Then, we recall that  $x$  is a monodromy-preserving deformation, therefore the monodromy matrices of (37) are independent of  $x$ . At  $\lambda = 0, 1, \infty$ , they are

$$M_0^{\text{OUT}} = \begin{Bmatrix} M_x M_0 \\ M_0 M_x \end{Bmatrix}, \quad M_1^{\text{OUT}} = M_1, \quad M_\infty^{\text{OUT}} = M_\infty.$$



Thus,  $\det(M_0^{\text{OUT}}) = 1$ , because  $\det(M_x) = \det(M_0) = 1$ . Therefore, there exists a constant matrix  $D$  and a complex constant number  $\sigma$  such that

$$D^{-1}M_0^{\text{OUT}}D = \begin{cases} \text{diag}(\exp\{-i\pi\sigma\}, \exp\{i\pi\sigma\}), \\ \begin{pmatrix} \pm 1 & * \\ 0 & \pm 1 \end{pmatrix}, & \text{or} \begin{pmatrix} \pm 1 & 0 \\ * & \pm 1 \end{pmatrix}, \end{cases} \quad \sigma \in \mathbf{Z}.$$

We conclude that  $\tilde{\sigma}(x) \equiv \sigma$ . We also have  $\text{tr}(M_0^{\text{OUT}}) = 2 \cos(\pi\sigma)$ .

Now consider the gauge

$$\Phi_1 := \lambda^{-\frac{\sigma}{2}}(\lambda - 1)^{-\frac{\theta_1}{2}} \Psi_{\text{OUT}}, \quad \frac{d\Phi_1}{d\lambda} = \left[ \frac{\hat{A}_0 + \hat{A}_x - \frac{\sigma}{2}}{\lambda} + \frac{\hat{A}_1 - \frac{\theta_1}{2}}{\lambda - 1} \right] \Phi_1. \tag{40}$$

We can identify  $\hat{A}_0 + \hat{A}_x - \frac{\sigma}{2}$  and  $\hat{A}_1 - \frac{\theta_1}{2}$  with  $B_0$  and  $B_1$  of proposition 2 in appendix A, case (A.1), with  $a = \frac{\theta_\infty}{2} + \frac{\theta_1}{2} + \frac{\sigma}{2}$ ,  $b = -\frac{\theta_\infty}{2} + \frac{\theta_1}{2} + \frac{\sigma}{2}$ ,  $c = \sigma$ .  $\square$

**Remark.**  $r_1$  may be a function of  $x$ . If the monodromy of system (37) depends on  $r_1$ , then  $r_1$  is a constant independent of  $x$ . This is the case here, but we do not prove it for reasons of space. See the references in part I.

Lemma 1 (and lemma 3 which follows) includes all cases (A.1)–(A.5) for system (40). Cases (A.2)–(A.5) are obtained substituting  $\sigma = -(\theta_\infty + \theta_1)$ ,  $\theta_\infty - \theta_1$ ,  $\theta_\infty + \theta_1$ ,  $\theta_1 - \theta_\infty$ , respectively. For all the computations which follow, involving system (37) or (40), we note that the hypothesis  $\theta_\infty \neq 0$  excludes cases (A.6), (A.7) and the Jordan cases (A.8)–(A.10).

The reduction to the Fuchsian system (11) is possible for  $x \rightarrow 0$  in the domain (6) if

$$\left| \frac{(A_0)_{ij}}{\lambda} + \frac{(A_x)_{ij}}{\lambda - x} \right| \gg |(A_1)_{ij}|, \quad \text{namely,} \quad \left| \frac{(A_0 + A_x)_{ij}}{x^{\delta_{\text{IN}}}} \right| \gg |(A_1)_{ij}|. \tag{41}$$

We can rewrite (11) using just the leading terms of the matrices:

$$\frac{d\Psi_{\text{IN}}}{d\lambda} = \left[ \frac{\hat{A}_0}{\lambda} + \frac{\hat{A}_x}{\lambda - x} \right] \Psi_{\text{IN}}. \tag{42}$$

Then, we re-scale  $\lambda$  and consider the following system:

$$\frac{d\Psi_{\text{IN}}}{d\mu} = \left( \frac{\hat{A}_0}{\mu} + \frac{\hat{A}_x}{\mu - 1} \right) \Psi_{\text{IN}}, \quad \mu := \frac{\lambda}{x}.$$

We know that there exists a matrix  $K_0(x)$  such that

$$K_0^{-1}(x)(\hat{A}_0 + \hat{A}_x)K_0(x) = \begin{pmatrix} \frac{\sigma}{2} & 0 \\ 0 & -\frac{\sigma}{2} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Let  $\hat{A}_i := K_0^{-1}\hat{A}_iK_0$ ,  $i = 0, x$ . By a gauge transformation, we get the system

$$\Psi_{\text{IN}} =: K_0(x)\Psi_0, \quad \frac{d\Psi_0}{d\mu} = \left[ \frac{\hat{A}_0}{\mu} + \frac{\hat{A}_x}{\mu - 1} \right] \Psi_0. \tag{43}$$

**Lemma 2.** Let  $r \in \mathbf{C}$ ,  $r \neq 0$ . If  $\sigma \neq 0$ , we have

$$\hat{A}_0 = \begin{pmatrix} \frac{\theta_0^2 - \theta_x^2 + \sigma^2}{4\sigma} & r \\ -\frac{[\sigma^2 - (\theta_0 - \theta_x)^2][\sigma^2 - (\theta_0 + \theta_x)^2]}{16\sigma^2} \frac{1}{r} & -\frac{\theta_0^2 - \theta_x^2 + \sigma^2}{4\sigma} \end{pmatrix}, \tag{44}$$

$$\hat{A}_x = \begin{pmatrix} \frac{\sigma^2 + \theta_x^2 - \theta_0^2}{4\sigma} & -r \\ \frac{[\sigma^2 - (\theta_0 - \theta_x)^2][\sigma^2 - (\theta_0 + \theta_x)^2]}{16\sigma^2} \frac{1}{r} & -\frac{\sigma^2 + \theta_x^2 - \theta_0^2}{4\sigma} \end{pmatrix}. \tag{45}$$

**Proof.** We do a gauge transformation:

$$\Phi_0 := \mu^{-\frac{\theta_0}{2}} (\mu - 1)^{-\frac{\theta_x}{2}} \Psi_0, \quad \frac{d\Phi_0}{d\mu} = \left[ \frac{\hat{A}_0 - \frac{\theta_0}{2}}{\mu} + \frac{\hat{A}_x - \frac{\theta_x}{2}}{\mu - 1} \right] \Phi_0. \quad (46)$$

We identify  $\hat{A}_0 - \frac{\theta_0}{2}, \hat{A}_x - \frac{\theta_x}{2}$  with  $B_0$  and  $B_1$  in appendix A, proposition 2, case (A.1), with  $a = \frac{\theta_0}{2} + \frac{\theta_x}{2} - \frac{\sigma}{2}, b = \frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\sigma}{2}, c = \theta_0$ .  $\square$

**Remark.** If the monodromy of system (43) depends on  $r$ , then  $r$  is a constant independent of  $x$ . This is the case here.

Lemma 2 (and lemma 4 which follows) includes also cases (A.2)–(A.5) for system (46). These cases correspond respectively to the values  $\sigma = \theta_0 + \theta_x, -\theta_0 - \theta_x, \theta_x - \theta_0, \theta_0 - \theta_x$ , with  $\theta_0 \neq \pm\theta_x$ .

4.1. Matching for  $\sigma \notin \mathbf{Z}$  and proof of (24)

We match  $\Psi_{\text{OUT}}$  and  $\Psi_{\text{IN}}$  in the intersection of the ‘outside’ and ‘inside’ domains, namely the region  $|x|^{\delta_{\text{OUT}}} \leq |\lambda| \leq |x|^{\delta_{\text{IN}}}, x \rightarrow 0$ . As a consequence, we obtain the leading term of  $y(x)$ .

**Lemma 3.** If  $\sigma \notin \mathbf{Z}$  and  $\theta_\infty \neq 0$ , system (37) has a fundamental matrix solution  $\Psi_{\text{OUT}}(\lambda)$  with the following behaviour at  $\lambda = 0$ :

$$\Psi_{\text{OUT}} = \sum_{n=0}^{\infty} G_n \lambda^n \begin{pmatrix} \lambda^{\frac{\sigma}{2}} & 0 \\ 0 & \lambda^{-\frac{\sigma}{2}} \end{pmatrix}, \quad G_0 = \begin{pmatrix} 1 & 1 \\ \frac{(\theta_\infty + \sigma)^2 - \theta_1^2}{4\theta_\infty r_1} & \frac{(\theta_\infty - \sigma)^2 - \theta_1^2}{4\theta_\infty r_1} \end{pmatrix}.$$

$G_n$  are matrices which depend rationally on  $\theta_\infty, \theta_1, \sigma, r_1$ . The series is convergent for  $|\lambda| < 1$ .

**Proof.** It is an immediate consequence of the standard theory of linear systems of Fuchsian differential equations.  $\square$

**Lemma 4.** If  $\sigma \notin \mathbf{Z}$ , system (43) has a fundamental matrix solution with the following behaviour at  $\mu = \infty$ :

$$\Psi_0(\mu) = \left[ I + \sum_{n=1}^{\infty} K_n \mu^{-n} \right] \begin{pmatrix} \mu^{\frac{\sigma}{2}} & 0 \\ 0 & \mu^{-\frac{\sigma}{2}} \end{pmatrix},$$

where  $I$  is the identity matrix,  $K_n$  are the matrices which depend rationally on  $\theta_0, \theta_x, \sigma, r$ . The series is convergent for  $|\mu| > 1$ .

**Proof.** It is a consequence of the standard theory of systems of Fuchsian equations.  $\square$

The matching relation  $\Psi_1(\lambda) \sim K_0(x)\Psi_0(\lambda/x), |x|^{\delta_{\text{OUT}}} \leq |\lambda| \leq |x|^{\delta_{\text{IN}}}, x \rightarrow 0$ , is

$$G_0 \begin{pmatrix} \lambda^{\frac{\sigma}{2}} & 0 \\ 0 & \lambda^{-\frac{\sigma}{2}} \end{pmatrix} \sim K_0(x) \begin{pmatrix} \lambda^{\frac{\sigma}{2}} & 0 \\ 0 & \lambda^{-\frac{\sigma}{2}} \end{pmatrix} \begin{pmatrix} x^{-\frac{\sigma}{2}} & 0 \\ 0 & x^{\frac{\sigma}{2}} \end{pmatrix}.$$

This gives the result

$$K_0(x) \sim \begin{pmatrix} 1 & 1 \\ \frac{(\theta_\infty + \sigma)^2 - \theta_1^2}{4\theta_\infty r_1} & \frac{(\theta_\infty - \sigma)^2 - \theta_1^2}{4\theta_\infty r_1} \end{pmatrix} \begin{pmatrix} x^{\frac{\sigma}{2}} & 0 \\ 0 & x^{-\frac{\sigma}{2}} \end{pmatrix}.$$

We compute the matrices  $\hat{A}_0(x) = K_0(x)\hat{A}_0K_0(x)^{-1}$ ,  $\hat{A}_x(x) = K_0(x)\hat{A}_xK_0(x)^{-1}$ , making use of lemma 2. We obtain

$$\hat{A}_0(x) = G_0 \begin{pmatrix} \frac{\theta_0^2 - \theta_x^2 + \sigma^2}{4\sigma} & rx^\sigma \\ -\frac{(\sigma + \theta_x - \theta_0)(\sigma + \theta_x + \theta_0)(\sigma - \theta_x + \theta_0)(\sigma - \theta_x - \theta_0)}{16\sigma^2 r} x^{-\sigma} & -\frac{\theta_0^2 - \theta_x^2 + \sigma^2}{4\sigma} \end{pmatrix} G_0^{-1}$$

$$\hat{A}_x(x) = G_0 \begin{pmatrix} \frac{\sigma^2 + \theta_x^2 - \theta_0^2}{4\sigma} & -rx^\sigma \\ \frac{(\sigma + \theta_x - \theta_0)(\sigma + \theta_x + \theta_0)(\sigma - \theta_x + \theta_0)(\sigma - \theta_x - \theta_0)}{16\sigma^2 r} x^{-\sigma} & -\frac{\sigma^2 + \theta_x^2 - \theta_0^2}{4\sigma} \end{pmatrix} G_0^{-1}.$$

This result shows that the matrix elements of  $\hat{A}_0$  and  $\hat{A}_x$  diverge as  $|x|^{-|\operatorname{Re} \sigma|}$  when  $x \rightarrow 0$  inside a sector (i.e., for  $|\arg(x)|$  bounded). In particular, we find  $(\hat{A}_1)_{12} = -r_1$  and

$$(\hat{A}_0)_{12} = \frac{r_1}{r} \frac{[\sigma^2 - (\theta_0 + \theta_x)^2][(\theta_0 - \theta_x)^2 - \sigma^2]}{16\sigma^3} x^{-\sigma} + r_1 \frac{\theta_0^2 - \theta_x^2 + \sigma^2}{2\sigma^2} - \frac{rr_1}{\sigma} x^\sigma.$$

The above are enough to compute the leading term(s) of  $y(x)$  from the formula

$$y(x) = \frac{x(A_0)_{12}}{x[(A_0)_{12} + (A_1)_{12}] - (A_1)_{12}} = -\frac{x(A_0)_{12}}{(A_1)_{12}} \left[ 1 - x \left( 1 + \frac{(A_0)_{12}}{(A_1)_{12}} \right) \right]^{-1}. \quad (47)$$

Thus,

$$y(x) \sim -x \frac{(\hat{A}_0)_{12}}{(\hat{A}_1)_{12}} = \left[ \frac{1}{r} \frac{[\sigma^2 - (\theta_0 + \theta_x)^2][(\theta_0 - \theta_x)^2 - \sigma^2]}{16\sigma^3} x^{1-\sigma} + \frac{\theta_0^2 - \theta_x^2 + \sigma^2}{2\sigma^2} x - \frac{r}{\sigma} x^{1+\sigma} \right]. \quad (48)$$

We have ignored  $1/[1 - x(1 + (\hat{A}_0)_{12}/(\hat{A}_1)_{12})]$  because condition (36) is equivalent to  $|x^{1-\delta_{1\pm\sigma}}| \rightarrow 0$  for  $x \rightarrow 0$ , which implies that  $|x(\hat{A}_0)_{12}/(\hat{A}_1)_{12}| \sim |x^{1\pm\sigma}| \rightarrow 0$ . Therefore,  $1/[1 - x(1 + (\hat{A}_0)_{12}/(\hat{A}_1)_{12})] = 1/(1 + O(x)) = 1 + O(x)$ .

If  $\operatorname{Re} \sigma \neq 0$ , the leading term of (48) is certainly correct, but some higher order corrections may be bigger than the next two terms of (48). If  $\operatorname{Re} \sigma = 0$ , the three terms of (48) are of the same order and their combination gives the trigonometric expression in theorem 2.

#### 4.2. Range of $\sigma$

Conditions (36) and (41) must be verified. Let  $C$  denote a non-zero constant. We suppose that  $x \rightarrow 0$  inside a sector with centre at  $x = 0$ . Then,

$$\text{Condition (41) is } |x|^{-\delta_{\text{IN}}} \gg C \iff \delta_{\text{IN}} > 0.$$

$$\text{Condition (36) is } C \gg |x|^{-|\operatorname{Re} \sigma|+1-\delta_{\text{OUT}}} \iff |\operatorname{Re} \sigma| < 1 - \delta_{\text{OUT}}.$$

The last condition implies that  $|\operatorname{Re} \sigma| < 1$ . We also conclude that  $0 < \delta_{\text{IN}} \leq \delta_{\text{OUT}} < 1$ .

#### 4.3. Leading term for $\sigma = \pm(\theta_0 + \theta_x)$ , $\pm(\theta_0 - \theta_x) \neq 0$ : proof of (25) and (26)

Formula (48) holds for any  $\sigma \neq 0$  such that  $|\operatorname{Re} \sigma| < 1$ . However, we cannot naively substitute the value of  $\sigma = \pm(\theta_0 + \theta_x)$ ,  $\pm(\theta_0 - \theta_x)$ , for which the coefficient of  $x^{1-\sigma}$  vanishes. This is because only the leading term is certainly correct, and it may be the term in  $x^{1-\sigma}$ . Therefore, here we briefly give the explicit derivation of (25) and (26), using cases (A.2)–(A.5) for system (46).

Case (A.2),  $a = 0$ . This is the case  $\sigma = \theta_0 + \theta_x \neq 0$ . The matching procedure does not change. From (A.2) we compute

$$\hat{A}_0 = G_0 \begin{pmatrix} \frac{\theta_0}{2} & rx^\sigma \\ 0 & -\frac{\theta_0}{2} \end{pmatrix} G_0^{-1}, \quad \hat{A}_x = G_0 \begin{pmatrix} \frac{\theta_x}{2} & -rx^\sigma \\ 0 & -\frac{\theta_x}{2} \end{pmatrix} G_0^{-1}.$$

This implies that  $(\hat{A}_0)_{12} = r_1 \left( \frac{\theta_0}{\theta_0 + \theta_x} - \frac{r}{\theta_0 + \theta_x} x^\sigma \right)$ , while  $(\hat{A}_1)_{12} = -r_1$  as in the generic case. Therefore,

$$y(x) \sim \frac{\theta_0}{\theta_0 + \theta_x} x - \frac{r}{\theta_0 + \theta_x} x^{\sigma+1}.$$

It is interesting to note that for  $\text{Re } \sigma > 0$  we have  $\hat{y}(x) \sim \theta_0 / (\theta_0 + \theta_x) x$ . Such a behaviour is what one would naively expect from the generic behaviour (48) when  $\sigma = 0$ .

Case (A.3),  $b = 0$ , is  $\sigma = -\theta_0 - \theta_x \neq 0$ . Case (A.4),  $a = c$ , is  $\sigma = \theta_x - \theta_0$ . Case (A.5),  $b = c$ , is  $\sigma = \theta_0 - \theta_x$ . Proceeding as above, we find (25) and (26).

**Remark.** If we substitute  $y = b_1x + b_2x^2 + b_3x^3 + \dots$  into (PVI) we find all the coefficients  $b_n$  by identifying equal powers of  $x$ . The result is (20). We need to assume that  $\theta_0 \pm \theta_x$  is not integer or zero.

4.4. Matching for  $\sigma = 0$ : proof of (27)

4.5. Case  $\theta_0 \pm \theta_x \neq 0$

**Lemma 5.** Let  $r_1 \in \mathbf{C}, r_1 \neq 0$ . The matrices of system (37) are

$$\hat{A}_1 = \begin{pmatrix} -\frac{\theta_\infty^2 + \theta_1^2}{4\theta_\infty} & -r_1 \\ \frac{[\theta_1^2 - \theta_\infty^2]^2}{16\theta_\infty^2 r_1} & \frac{\theta_\infty^2 + \theta_1^2}{4\theta_\infty} \end{pmatrix}, \quad \hat{A}_0 + \hat{A}_x = \begin{pmatrix} \frac{\theta_1^2 - \theta_\infty^2}{4\theta_\infty} & r_1 \\ -\frac{[\theta_\infty^2 - \theta_1^2]^2}{16\theta_\infty^2 r_1} & \frac{\theta_\infty^2 - \theta_1^2}{4\theta_\infty} \end{pmatrix}, \quad \forall r_1 \neq 0.$$

A fundamental matrix solution can be chosen with the following behaviour at  $\lambda = 0$ :

$$\Psi_{\text{OUT}}(\lambda) = [G_0 + O(\lambda)] \begin{pmatrix} 1 & \log \lambda \\ 0 & 1 \end{pmatrix}, \quad G_0 = \begin{pmatrix} 1 & 0 \\ \frac{\theta_\infty^2 - \theta_1^2}{4\theta_\infty r_1} & r_1 \end{pmatrix}.$$

**Proof.** System (40) is

$$\frac{d\Phi_1}{d\lambda} = \left[ \frac{\hat{A}_0 + \hat{A}_x}{\lambda} + \frac{\hat{A}_1 - \frac{\theta_1}{2}}{\lambda - 1} \right] \Phi_1.$$

We identify  $\hat{A}_0 + \hat{A}_x$  and  $\hat{A}_1 - \frac{\theta_1}{2}$  with  $B_0$  and  $B_1$  of proposition 2 in appendix A, diagonalizable cases (A.1)–(A.5) (we recall that (A.6)–(A.10) never occur when  $\theta_\infty \neq 0$ ) with  $a = \frac{\theta_\infty}{2} + \frac{\theta_1}{2}, b = -\frac{\theta_\infty}{2} + \frac{\theta_1}{2}, c = 0$ .

The behaviour of a fundamental solution is a standard result in the theory of Fuchsian systems. The matrix  $G_0$  is defined by  $G_0^{-1} (\hat{A}_0 + \hat{A}_x) G_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .  $\square$

**Lemma 6.** Let  $r \in \mathbf{C}$ . The matrices of system (43) are

$$\hat{A}_0 = \begin{pmatrix} r + \frac{\theta_0}{2} & \frac{4r(r+\theta_0)}{\theta_x^2 - \theta_0^2} \\ \frac{\theta_0^2 - \theta_x^2}{4} & -r - \frac{\theta_0}{2} \end{pmatrix}, \quad \hat{A}_x = \begin{pmatrix} -r - \frac{\theta_0}{2} & 1 - \frac{4r(r+\theta_0)}{\theta_x^2 - \theta_0^2} \\ \frac{\theta_x^2 - \theta_0^2}{4} & r + \frac{\theta_0}{2} \end{pmatrix}. \quad (49)$$

There exist a fundamental solution of (43) with the following behaviour at  $\mu = \infty$ :

$$\Psi_0(\mu) = \left[ I + O\left(\frac{1}{\mu}\right) \right] \begin{pmatrix} 1 & \log \mu \\ 0 & 1 \end{pmatrix}, \quad \mu \rightarrow \infty.$$

**Proof.** To compute  $\hat{A}_0$  and  $\hat{A}_x$  for the generic case, we consider case (A.8) in proposition 2, applied to the system (46). The parameters are  $a = \frac{\theta_0}{2} + \frac{\theta_x}{2}$ ,  $c = \theta_0$ . In particular,

$$\hat{A}_0 - \frac{\theta_0}{2} + \hat{A}_x - \frac{\theta_x}{2} = \begin{pmatrix} -\frac{\theta_0+\theta_x}{2} & 1 \\ 0 & -\frac{\theta_0+\theta_x}{2} \end{pmatrix}. \tag{50}$$

Here, the values of the parameters satisfy the conditions  $a \neq 0$  and  $a \neq c$ , namely  $\theta_0 \pm \theta_x \neq 0$ . From the matrices (A.8), we obtain  $\hat{A}_0 = B_0 + \theta_0/2$  and  $\hat{A}_x = B_1 + \theta_x/2$ . Keeping into account (50), by the standard theory of Fuchsian systems we have

$$\Phi_0(\mu) = \left[ I + O\left(\frac{1}{\mu}\right) \right] \mu^{-\frac{\theta_0+\theta_x}{2}} \begin{pmatrix} 1 & \log \mu \\ 0 & 1 \end{pmatrix}, \quad \mu \rightarrow \infty.$$

This proves the behaviour of  $\Psi_0(\mu)$ . □

The matching condition  $\Psi_{\text{OUT}}(\lambda) \sim K_0(x)\Psi_0(\lambda/x)$  becomes

$$K_0(x) \begin{pmatrix} 1 & \log\left(\frac{\lambda}{x}\right) \\ 0 & 1 \end{pmatrix} \sim G_0 \begin{pmatrix} 1 & \log \lambda \\ 0 & 1 \end{pmatrix} \implies K_0(x) \sim \begin{pmatrix} 1 & 0 \\ \frac{\theta_\infty^2 - \theta_1^2}{4\theta_\infty r_1} & \frac{1}{r_1} \end{pmatrix} \begin{pmatrix} 1 & \log x \\ 0 & 1 \end{pmatrix}.$$

From the above result, together with (49), we compute  $\hat{A}_0 = K_0 \hat{A}_0 K_0^{-1}$ ,  $\hat{A}_1 = K_0 \hat{A}_1 K_0^{-1}$ . For example,

$$\hat{A}_0 = G_0 \begin{pmatrix} r + \frac{\theta_0}{2} + \frac{\theta_0^2 - \theta_x^2}{4} \log x & \frac{\theta_x^2 - \theta_0^2}{4} \log^2 x - 2\left(r + \frac{\theta_0}{2}\right) \log x + \frac{4r(r+\theta_0)}{\theta_x^2 - \theta_0^2} \\ \frac{\theta_0^2 - \theta_x^2}{4} & \frac{\theta_x^2 - \theta_0^2}{4} \log x - \left(r + \frac{\theta_0}{2}\right) \end{pmatrix} G_0^{-1}.$$

A similar expression holds for  $\hat{A}_x$ . The reader can verify that the matching conditions (36), (41) are satisfied.

The leading terms of  $y(x)$  are obtained from (47) with matrix entries  $(\hat{A}_1)_{12} = -r_1$  and

$$(\hat{A}_0)_{12} = r_1 \left[ \frac{\theta_x^2 - \theta_0^2}{4} \log^2 x - 2\left(r + \frac{\theta_0}{2}\right) \log x + \frac{4r(r+\theta_0)}{\theta_x^2 - \theta_0^2} \right].$$

The result is

$$y(x) \sim x \left[ \frac{\theta_x^2 - \theta_0^2}{4} \log^2 x - 2\left(r + \frac{\theta_0}{2}\right) \log x + \frac{4r(r+\theta_0)}{\theta_x^2 - \theta_0^2} \right]. \tag{51}$$

#### 4.6. Case $\theta_0 \pm \theta_x = 0$

We consider here cases (A.9), (A.10) of proposition 2 applied to system (46).

Case (A.9) is the case  $\sigma = 0$ ,  $\theta_0 = -\theta_x$ , with  $a = 0$ ,  $c = \theta_0$  in system (46). From proposition 2, we immediately have

$$\hat{A}_0 = \begin{pmatrix} \frac{\theta_0}{2} & r \\ 0 & -\frac{\theta_0}{2} \end{pmatrix}, \quad \hat{A}_x = \begin{pmatrix} \frac{\theta_x}{2} & 1-r \\ 0 & -\frac{\theta_x}{2} \end{pmatrix}.$$

The behaviour of  $\Psi_0$  and  $\Psi_{\text{OUT}}$ , and the matching are the same as subsection 4.5. We obtain the same  $K_0(x)$ . Therefore,

$$(\hat{A}_0)_{12} = r_1(r - \theta_0 \ln x), \quad (\hat{A}_1)_{12} = -r_1.$$

This gives the leading terms:

$$y(x) \sim x(r - \theta_0 \ln x) = x(r + \theta_x \ln x). \tag{52}$$

In the same way, we treat the other cases. Case (A.9), with  $a = c$ , is the case  $\sigma = 0$ ,  $\theta_0 = \theta_x$ . As above, we find  $y(x) \sim x(r - \theta_0 \ln x) = x(r - \theta_x \ln x)$ . Case (A.10),

with  $a = 0$ , is the case  $\sigma = 0, \theta_0 = -\theta_x$ . We find  $y(x) \sim x(r + \theta_0 \ln x) = x(r - \theta_x \ln x)$ . Case (A.10), with  $a = c$ , is the case  $\sigma = 0, \theta_0 = \theta_x$ . We find  $y(x) \sim x(r + \theta_0 \ln x) = x(r + \theta_x \ln x)$ .

Both (51) and (52) contain more than one term, and in principle only the leading one is certainly correct. To prove that they are all correct, we observe that (51) and (52) can also be obtained by direct substitution of  $y(x) = x(A_1 + B_1 \ln x + C_1 \ln^2 x + D_1 \ln^3 x + \dots) + x^2(A_2 + B_2 \ln x + \dots) + \dots$  into (PVI). We can recursively determine the coefficients by identifying the same powers of  $x$  and  $\ln x$ . As a result, we obtain only the five cases (28), which include (51) and (52).

4.7. No naive matching for  $\sigma = 1$

The condition  $|\operatorname{Re} \sigma| < 1$  suggests that the matching above does not work in the case  $\sigma = 1$  (and  $\sigma = -1$ , being equivalent). Let us convince ourselves of this fact by repeating the procedure above. A fundamental matrix solution for (37) at  $\lambda = 0$  is non-generic:

$$\Psi_{\text{OUT}}(\lambda) = (G_0 + O(\lambda)) \begin{pmatrix} \lambda^{\frac{1}{2}} & 0 \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \log \lambda \\ 0 & 1 \end{pmatrix}, \tag{53}$$

where

$$G_0 = \begin{pmatrix} 1 & \frac{4}{\theta_1^2 - (\theta_\infty - 1)^2} \\ \frac{(\theta_\infty + 1)^2 - \theta_1^2}{4\theta_\infty r_1} & -\frac{1}{\theta_\infty r_1} \end{pmatrix}, \quad \forall r_1 \neq 0.$$

A fundamental matrix solution of (43) at  $\mu = \infty$  is non-generic:

$$\Psi_0(\mu) = \left( I + O\left(\frac{1}{\mu}\right) \right) \begin{pmatrix} \mu^{\frac{1}{2}} & 0 \\ 0 & \mu^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R \log \mu & 1 \end{pmatrix}, \tag{54}$$

where

$$R := (\hat{A}_x)_{21} = \frac{[(\theta_0 + \theta_x)^2 - 1][(\theta_0 - \theta_x)^2 - 1]}{16r}, \quad r \neq 0.$$

The matching relation,

$$K_0(x) \begin{pmatrix} \left(\frac{\lambda}{x}\right)^{\frac{1}{2}} & 0 \\ R \left(\frac{\lambda}{x}\right)^{-\frac{1}{2}} \log\left(\frac{\lambda}{x}\right) & \left(\frac{\lambda}{x}\right)^{-\frac{1}{2}} \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{4}{\theta_1^2 - (\theta_\infty - 1)^2} \\ \frac{(\theta_\infty + 1)^2 - \theta_1^2}{4\theta_\infty r_1} & -\frac{1}{\theta_\infty r_1} \end{pmatrix} \begin{pmatrix} \lambda^{\frac{1}{2}} & \lambda^{\frac{1}{2}} \log \lambda \\ 0 & \lambda^{-\frac{1}{2}} \end{pmatrix},$$

shows that we cannot eliminate  $\lambda$  to obtain  $K_0(x)$ .

One case  $\sigma = 1$  is studied in part III, making use of a non-Fuchsian reductions of system (1).

4.8. Monodromy data

Systems (37), (43) are equivalent to Gauss hyper-geometric equations, as it is explained in appendix A (make use of systems (40) and (46), respectively). Therefore, the monodromy can be computed in a standard way, using the connection formulae for the hyper-geometric functions.

We obtain in this way the monodromy of  $\Psi_{\text{OUT}}$  and  $\Psi_{\text{IN}}$ . As it is explained in section 2.2, it may be necessary to do a transformation  $\Psi_{\text{OUT}} \mapsto \Psi_{\text{OUT}}^{\text{Match}} := \Psi_{\text{OUT}} C_{\text{OUT}}$ , in order to match the ‘out’ and ‘in’ solutions with a solution  $\Psi$  of (1). In this way, the monodromy matrices  $M_0, M_x, M_1$  of  $\Psi$  can be obtained. They depend on  $r$ . We then compute the traces of  $M_i M_j$  and extract  $r$ , which is thus obtained as a function of the monodromy data.

We do not repeat the computations here. One example is the computation of (34) in [16] and [5, 8, 9, 11].

### Part III. Derivations for non-Fuchsian reduction

#### 5. Case $\theta_1 = \pm\theta_\infty$ , $\theta_x = \pm\theta_0$ , $\lim_{x \rightarrow 0}(A_x + A_0) = \mathbf{0}$ : solution (16)

We begin by observing that for  $\theta_x = \pm\theta_0$ , system (46) may fall in cases (A.6) and (A.7). If it is so, then  $\hat{A}_0 + \hat{A}_x = 0$ , and therefore  $\hat{A}_0 + \hat{A}_x = 0$ . More precisely, we start from the following hypotheses:

$$\lim_{x \rightarrow 0}(A_0(x) + A_x(x)) = 0, \quad A := \lim_{x \rightarrow 0} A_x(x) = \text{a constant matrix with eigenvalues } \pm \frac{\theta_x}{2}.$$

The first hypothesis means that we can write (the trace is zero)

$$A_0 + A_x = \begin{pmatrix} a(x) & b(x)r \\ c(x)\frac{1}{r} & -a(x) \end{pmatrix}, \quad \lim_{x \rightarrow 0} a(x) = \lim_{x \rightarrow 0} b(x) = \lim_{x \rightarrow 0} c(x) = 0.$$

The second hypotheses implies that the general form of  $A$  is

$$A = \begin{pmatrix} s + \frac{\theta_x}{2} & -r \\ \frac{(s+\theta_x)s}{r} & -s - \frac{\theta_x}{2} \end{pmatrix}, \quad r, s \in \mathbf{C}, \quad r \neq 0.$$

We also write

$$A_x(x) - A =: \Delta_x(x), \quad A_0 + A =: \Delta_0(x), \quad \Delta_0 + \Delta_x \equiv A_0 + A_x.$$

$\Delta_x(x)$  and  $\Delta_0(x)$  are vanishing. We suppose that the slowest vanishing behaviour be of order  $x^{\sigma_0}$ , for some  $\sigma_0 > 0$ . Namely,

$$a(x), b(x), c(x), (\Delta_x)_{ij}(x), (\Delta_0)_{ij}(x) = O(x^{\sigma_0}), \quad \sigma_0 > 0.$$

Finally, we have

$$A_1(x) = -\frac{\theta_\infty}{2}\sigma_3 - (A_0 + A_x) \longrightarrow -\frac{\theta_\infty}{2}\sigma_3, \quad x \rightarrow 0.$$

#### 5.1. Coalescence of singularities

(1) *The system for  $\Psi_{\text{OUT}}$*

We consider system (8), in the domain  $|\lambda| \geq |x|^{\delta_{\text{OUT}}}$ . Let us determine the conditions to neglect a term  $x^n A_x / \lambda^{n+1}$ —and all the terms following it—with respect to  $(A_0 + A_x) / \lambda$ , when  $x \rightarrow 0$ ,  $\lambda \sim x^\delta$ ,  $\delta \leq \delta_{\text{OUT}}$ .

$$\text{We can neglect } \frac{x^n A_x}{\lambda^{n+1}} \iff \left| \frac{x^n}{\lambda^n} A_x \right| \ll |(A_0 + A_x)_{ij}|, \quad \forall i, j \in \{1, 2\}.$$

Since  $\lim_{x \rightarrow 0}(A_x)_{ij}$  are non-zero constants, the above condition is  $|x|^{n-n\delta} \ll |x|^{\sigma_0}$ , namely,  $\delta < 1 - \sigma_0/n$ . We state this result as a lemma.

**Lemma 7.** *Let  $N_{\text{OUT}} \geq 2$  be an integer. We can approximate (8) with*

$$\frac{d\Psi_{\text{OUT}}}{d\lambda} = \left[ \frac{(A_0 + A_x)}{\lambda} + \frac{A_x}{\lambda} \sum_{n=1}^{N_{\text{OUT}}-1} \left(\frac{x}{\lambda}\right)^n + \frac{A_1}{\lambda - 1} \right] \Psi_{\text{OUT}},$$

*if and only if*

$$\delta_{\text{OUT}} < 1 - \frac{\sigma_0}{N_{\text{OUT}}}. \quad (55)$$

Suppose that all term  $x^n A_x / \lambda^{n+1}$ , with  $n \geq N_{OUT}$ , have been neglected. We can also make the substitution  $A_1 \mapsto -\frac{\theta_\infty}{2} \sigma_3$  in  $\frac{A_1}{\lambda-1}$ , if and only if the error term  $-\frac{A_0+A_x}{\lambda-1}$  is smaller than  $\frac{x^{N_{OUT}-1} A_x}{\lambda^{N_{OUT}}}$ . Namely, if and only if

$$|(A_0 + A_x)_{ij}| \ll \left| \frac{x^{N_{OUT}-1} (A_x)_{ij}}{\lambda^{N_{OUT}}} \right|.$$

This is  $|x|^{\sigma_0} \ll |x|^{N_{OUT}-1-N_{OUT}\delta_{OUT}}$ , namely  $\delta_{OUT} > 1 - \frac{1+\sigma_0}{N_{OUT}}$ .

We can also do the substitution  $A_x \mapsto A$ , provided that  $\delta_{OUT} < 1$ . This is because we can neglect terms  $\frac{x^n \Delta_x}{\lambda^{n+1}}$  with respect to  $\frac{A_0+A_x}{\lambda}$ , where both  $\Delta_x$  and  $A_0 + A_x$  are  $O(x^{\sigma_0})$ ,  $\lambda \sim x^\delta$ ,  $\delta < 1$ . We summarize the result in the following lemma.

**Lemma 8.** *Let  $N_{OUT} \geq 2$  be an integer. We can approximate (8) with*

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[ \frac{A_0 + A_x}{\lambda} + \frac{A}{\lambda} \sum_{n=1}^{N_{OUT}-1} \left(\frac{x}{\lambda}\right)^n - \frac{\theta_\infty}{2} \frac{\sigma_3}{\lambda-1} \right] \Psi_{OUT},$$

if and only if

$$1 - \frac{1 + \sigma_0}{N_{OUT}} < \delta_{OUT} < 1 - \frac{\sigma_0}{N_{OUT}}.$$

In particular, this means that  $\delta_{OUT} < 1$ .

**Example.** If  $\sigma_0 = 1$  and  $N_{IN} = 2$  we have

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[ \frac{x A}{\lambda^2} + \frac{A_0 + A_x}{\lambda} - \frac{\theta_\infty}{2} \frac{\sigma_3}{\lambda-1} \right] \Psi_{OUT}, \quad 0 < \delta_{OUT} < \frac{1}{2}.$$

If  $\sigma_0 = 1$  and  $N_{IN} = 3$  we have

$$\frac{d\Psi_{OUT}}{d\lambda} = \left[ \frac{x^2 A}{\lambda^3} + \frac{x A}{\lambda^2} + \frac{A_0 + A_x}{\lambda} - \frac{\theta_\infty}{2} \frac{\sigma_3}{\lambda-1} \right] \Psi_{OUT}, \quad \frac{1}{3} < \delta_{OUT} < \frac{2}{3}.$$

(2) *The system for  $\Psi_{IN}$*

We consider system (9) in the domain  $|\lambda| \leq |x|^{\delta_{IN}}$ . We investigate the condition necessary and sufficient to neglect a term  $\lambda^n A_1$  (and all its next terms) with respect to  $\frac{A_0}{\lambda} + \frac{A_x}{\lambda-x}$ . It is convenient to write

$$\frac{A_0}{\lambda} + \frac{A_x}{\lambda-x} = \frac{A_0 + A_x}{\lambda-x} - \frac{x A_0}{\lambda(\lambda-x)}.$$

Suppose that  $\lambda \sim x^\delta$ ,  $\delta \geq \delta_{IN}$ .

$$\text{We neglect } A_1 \lambda^n \iff \begin{cases} \left| \frac{x A_0}{\lambda(\lambda-x)} \right| \gg |A_1 \lambda^n|, \text{ namely, } |x|^{1-2\delta} \gg |x|^{n\delta} & \Leftrightarrow \delta > \frac{1}{n+2}; \\ \left| \frac{A_0+A_x}{\lambda-x} \right| \gg |A_1 \lambda^n|, \text{ namely, } |x|^{\sigma_0-\delta} \gg |x|^{n\delta} & \Leftrightarrow \delta > \frac{\sigma_0}{n+1}. \end{cases}$$

Thus, we have the condition  $\delta > \max \left\{ \frac{\sigma_0}{n+1}, \frac{1}{n+2} \right\}$ . We have proven the following:

**Lemma 9.** *Let  $N_{IN} \geq 1$  be an integer. We approximate (9) with*

$$\frac{d\Psi_{IN}}{d\lambda} = \left[ \frac{A_0}{\lambda} + \frac{A_x}{\lambda-x} - A_1 \sum_{n=0}^{N_{IN}-1} \lambda^n \right] \Psi_{IN},$$

if and only if

$$\delta_{IN} > \max \left\{ \frac{\sigma_0}{N_{IN} + 1}, \frac{1}{N_{IN} + 2} \right\}.$$



We further make the substitution  $A_1 \mapsto -\frac{\theta_\infty}{2}\sigma_3$ . This is possible if and only if two conditions are true: (1)  $|\frac{x A_0}{\lambda(\lambda-x)}|$  and  $|\frac{A_0+A_x}{\lambda-x}|$  are dominant w.r.t. the term  $A_0 + A_x$  appearing in  $A_1 = -\frac{\theta_\infty}{2}\sigma_3 - (A_0 + A_x)$ . (2)  $|\lambda^{N_{\text{IN}}-1} A_1|$ , i.e.  $|\lambda^{N_{\text{IN}}-1}\sigma_3|$ , is dominant w.r.t. the term  $A_0 + A_x$  in  $A_1$ . Explicitly, the conditions are

$$\begin{aligned} \left| \frac{A_0 + A_x}{\lambda - x} \right| &\gg |A_0 + A_x| \quad (\text{this is always true}), \\ \left| \frac{x A_0}{\lambda(\lambda - x)} \right| &\gg |A_0 + A_x| \iff |x|^{1-2\delta} > x^{\sigma_0}, \quad \text{namely, } \delta > \frac{1 - \sigma_0}{2}, \end{aligned}$$

and

$$|\lambda^{N_{\text{IN}}-1}\sigma_3| \gg |A_0 + A_x| \iff |x|^{(N_{\text{IN}}-1)\delta} > |x|^{\sigma_0}, \quad \text{namely, } \delta < \frac{\sigma_0}{N_{\text{IN}} - 1}.$$

We have

**Lemma 10.** *Let  $N_{\text{IN}} \geq 1$  be an integer. We approximate (9) with*

$$\begin{aligned} \frac{d\Psi_{\text{IN}}}{d\lambda} &= \left[ \frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{\theta_\infty}{2}\sigma_3 \sum_{n=0}^{N_{\text{IN}}-1} \lambda^n \right] \Psi_{\text{IN}} \\ &= \left[ \frac{A_0 + A_x}{\lambda - x} - \frac{x A_0}{\lambda(\lambda - x)} + \frac{\theta_\infty}{2}\sigma_3 \sum_{n=0}^{N_{\text{IN}}-1} \lambda^n \right] \Psi_{\text{IN}}, \end{aligned}$$

if and only if

$$\max \left\{ \frac{1 - \sigma_0}{2}, \frac{\sigma_0}{N_{\text{IN}} + 1}, \frac{1}{N_{\text{IN}} + 2} \right\} < \delta_{\text{IN}} < \frac{\sigma_0}{N_{\text{IN}} - 1}.$$

As a final simplification, we substitute  $A_0 = -A + \Delta_0 \mapsto -A$ . This is possible if and only if

$$\left| \frac{x \Delta_0}{\lambda(\lambda - x)} \right| \ll \left| \frac{A_0 + A_x}{\lambda - x} \right| \iff |x|^{1+\sigma_0-2\delta} < |x|^{\sigma_0-\delta}, \quad \text{namely, } \delta < 1,$$

and

$$\left| \frac{x \Delta_0}{\lambda(\lambda - x)} \right| \ll \left| \lambda^{N_{\text{IN}}-1} \frac{\theta_\infty}{2}\sigma_3 \right| \iff |x|^{1+\sigma_0-2\delta} \ll x^{(N_{\text{IN}}-1)\delta}, \quad \text{namely, } \delta < \frac{\sigma_0 + 1}{N_{\text{IN}} + 1}.$$

We have proven the following:

**Lemma 11.** *Let  $N_{\text{IN}} \geq 1$  be an integer. We can approximate (9) with*

$$\frac{d\Psi_{\text{IN}}}{d\lambda} = \left[ \frac{A_0 + A_x}{\lambda - x} + \frac{x A}{\lambda(\lambda - x)} + \frac{\theta_\infty}{2}\sigma_3 \sum_{n=0}^{N_{\text{IN}}-1} \lambda^n \right] \Psi_{\text{IN}},$$

if and only if

$$\max \left\{ \frac{1 - \sigma_0}{2}, \frac{\sigma_0}{N_{\text{IN}} + 1}, \frac{1}{N_{\text{IN}} + 2} \right\} < \delta_{\text{IN}} < \min \left\{ \frac{\sigma_0}{N_{\text{IN}} - 1}, \frac{\sigma_0 + 1}{N_{\text{IN}} + 1} \right\}.$$

**Examples.** If  $\sigma_0 = 1$  and  $N_{\text{IN}} = 1$ , we have

$$\frac{d\Psi_{\text{IN}}}{d\lambda} = \left[ \frac{A_0 + A_x}{\lambda - x} + \frac{x A}{\lambda(\lambda - x)} + \frac{\theta_\infty}{2}\sigma_3 \right] \Psi_{\text{IN}}, \quad \frac{1}{2} < \delta_{\text{IN}} < 1.$$

If we keep  $-A_0$  instead of  $A$ , with no change in the condition on  $\delta_{IN}$ , we can also rewrite

$$\frac{d\Psi_{IN}}{d\lambda} = \left[ \frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{\theta_\infty}{2} \sigma_3 \right] \Psi_{IN}.$$

If  $\sigma_0 = 1$  and  $N_{IN} = 2$ , we have

$$\frac{d\Psi_{IN}}{d\lambda} = \left[ \frac{A_0 + A_x}{\lambda - x} + \frac{x A}{\lambda(\lambda - x)} + \frac{\theta_\infty}{2} \sigma_3(1 + \lambda) \right] \Psi_{IN}, \quad \frac{1}{3} < \delta_{IN} < \frac{2}{3}.$$

Equivalently, we can write

$$\frac{d\Psi_{IN}}{d\lambda} = \left[ \frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{\theta_\infty}{2} \sigma_3(1 + \lambda) \right] \Psi_{IN}.$$

### 5.2. Matching

We do the matching in the overlapping region  $|x|^{\delta_{OUT}} \leq |\lambda| \leq |x|^{\delta_{IN}}$ . This imposes  $\delta_{IN} \leq \delta_{OUT}$ . In order for the overlapping region not to be empty, we must choose suitable reductions of (8) and (9). If we expect  $\sigma_0$  to be close to 1, we try to match solutions  $\Psi_{OUT}$  and  $\Psi_{IN}$  satisfying one of the following sets of systems:

*First choice:*

$$\begin{aligned} \frac{d\Psi_{OUT}}{d\lambda} &= \left[ \frac{x A}{\lambda^2} + \frac{A_0 + A_x}{\lambda} - \frac{\theta_\infty}{2} \frac{\sigma_3}{\lambda - 1} \right] \Psi_{OUT}, \\ \frac{d\Psi_{IN}}{d\lambda} &= \left[ \frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{\theta_\infty}{2} \sigma_3(1 + \lambda) \right] \Psi_{IN}. \end{aligned}$$

The condition to be satisfied for  $\sigma_0 \cong 1$  is  $\frac{\sigma_0}{3} < \delta_{IN} \leq \delta_{OUT} < 1 - \frac{\sigma_0}{2}$ . For  $\sigma_0 = 1$ , this is

$$\frac{1}{3} < \delta_{IN} \leq \delta_{OUT} < \frac{1}{2}.$$

*Second choice:*

$$\begin{aligned} \frac{d\Psi_{OUT}}{d\lambda} &= \left[ \frac{x^2 A}{\lambda^3} + \frac{x A}{\lambda^2} + \frac{A_0 + A_x}{\lambda} - \frac{\theta_\infty}{2} \frac{\sigma_3}{\lambda - 1} \right] \Psi_{OUT}, \\ \frac{d\Psi_{IN}}{d\lambda} &= \left[ \frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{\theta_\infty}{2} \sigma_3 \right] \Psi_{IN}. \end{aligned}$$

For  $\sigma_0 \cong 1$ , the condition to be satisfied is  $\frac{\sigma_0}{2} < \delta_{IN} \leq \delta_{OUT} < 1 - \frac{\sigma_0}{3}$ . For  $\sigma_0 = 1$ , this is

$$\frac{1}{2} < \delta_{IN} \leq \delta_{OUT} < \frac{2}{3}.$$

In both cases, the overlapping regions are not empty. The matching procedure will determine the leading terms (order  $x^{\sigma_0}$ ) of the unknown matrix elements  $a(x)$ ,  $b(x)$ ,  $c(x)$  of  $A_0 + A_x$ .

### 5.3. Matching for the first choice: $\frac{1}{3} < \delta_{IN} \leq \delta_{OUT} < \frac{1}{2}$

We rewrite the systems in a more convenient form:

$$\nu := \frac{1}{\lambda}, \quad \mu := \frac{\lambda}{x}; \quad \frac{d\Psi_{OUT}}{d\nu} = \left[ -x A - \frac{A_0 + A_x}{\nu} - \frac{\theta_\infty}{2} \frac{\sigma_3}{\nu(\nu - 1)} \right] \Psi_{OUT}; \quad (56)$$

$$\begin{aligned} \frac{d\Psi_{\text{IN}}}{d\mu} &= \left[ x^2 \frac{\theta_\infty}{2} \sigma_3 \mu + x \frac{\theta_\infty}{2} \sigma_3 + \frac{A_0}{\mu} + \frac{A_x}{\mu - 1} \right] \Psi_{\text{IN}} \\ &= \left[ x^2 \frac{\theta_\infty}{2} \sigma_3 \mu + x \frac{\theta_\infty}{2} \sigma_3 + \frac{A_0 + A_x}{\mu} - \frac{A_0}{\mu(\mu - 1)} \right] \Psi_{\text{IN}}. \end{aligned}$$

Then we substitute  $A_0 \mapsto -A$  in the last term.

In the matching region  $|x|^{-\delta_{\text{IN}}} \leq |v| \leq |x|^{-\delta_{\text{OUT}}}$ ,  $|x|^{\delta_{\text{OUT}}-1} \leq \mu \leq |x|^{\delta_{\text{IN}}-1}$ , we have  $v \rightarrow \infty, \mu \rightarrow \infty$ . The point at infinity is a non-Fuchsian singularity<sup>5</sup>. In order to find the local behaviour at this point, it is convenient to put the leading term in diagonal form. Let  $G$  be the invertible matrix such that

$$G^{-1}AG = -\frac{\theta_x}{2}\sigma_3, \quad \text{for example,} \quad G = \begin{pmatrix} 1 & \frac{r}{s} \\ \frac{s+\theta_x}{r} & 1 \end{pmatrix},$$

and put

$$\Psi_{\text{OUT}} =: G\tilde{\Psi}_{\text{OUT}}.$$

Then,

$$\frac{d\tilde{\Psi}_{\text{OUT}}}{dv} = \left[ x \frac{\theta_x}{2} \sigma_3 - \frac{G^{-1}(A_0 + A_x)G}{v} - \frac{\theta_\infty}{2} G^{-1} \sigma_3 G \left( \frac{1}{v^2} + \frac{1}{v^3} + \dots \right) \right] \tilde{\Psi}_{\text{OUT}}, \quad v \rightarrow \infty; \tag{57}$$

$$\frac{d\Psi_{\text{IN}}}{d\mu} = \left[ x^2 \frac{\theta_\infty}{2} \sigma_3 \mu + x \frac{\theta_\infty}{2} \sigma_3 + \frac{A_0 + A_x}{\mu} + A \left( \frac{1}{\mu^2} + \frac{1}{\mu^3} + \dots \right) \right] \Psi_{\text{IN}}, \quad \mu \rightarrow \infty. \tag{58}$$

In order to write the local behaviour of  $\tilde{\Psi}_{\text{OUT}}$  and  $\Psi_{\text{IN}}$  at infinity, we observe that systems (57) and (58), respectively, have the following forms:

$$\frac{dY_1}{dz} = \left[ \Omega + \frac{D_1}{z} + \frac{D_2}{z^2} + \frac{D_3}{z^3} + \dots \right] Y_1, \tag{59}$$

$$\frac{dY_2}{dz} = \left[ x^2 \Lambda z + x \Lambda z + \frac{E_1}{z} + \frac{E_2}{z^2} + \frac{E_3}{z^3} + \dots \right] Y_2, \tag{60}$$

where  $\Omega$  and  $\Lambda$  are the diagonal matrices with *distinct eigenvalues*. In our case,

$$\Omega = x \frac{\theta_x}{2} \sigma_3, \quad \Lambda = \frac{\theta_\infty}{2} \sigma_3.$$

The eigenvalues are distinct iff  $\theta_x \neq 0, \theta_\infty \neq 0$ .

The theory for such systems is developed in [3] (see also [4]). For any sector of angular width  $\pi + \epsilon, \epsilon > 0$  sufficiently small, there exists a unique solution of (59) with asymptotic expansion:

$$Y_1(z) \sim \left[ I + \frac{G_1}{z} + \frac{G_2}{z^2} + \dots \right] \exp\{\Omega z\} z^{\Omega_1}, \quad z \rightarrow \infty.$$

<sup>5</sup> System (56) can also be written with  $\frac{\theta_\infty}{2} \sigma_3 \mapsto -A_1$ :

$$\frac{d\Psi_{\text{OUT}}}{dv} = \left[ -xA - \frac{A_0 + A_x}{v} + \frac{A_1}{v(v-1)} \right] \Psi_{\text{OUT}} \left[ -xA - \frac{A_0 + A_x + A_1}{v} + \frac{A_1}{v-1} \right] \Psi_{\text{OUT}}.$$

After diagonalization, we get

$$\frac{d\tilde{\Psi}_{\text{OUT}}}{dv} = \left[ x \frac{\theta_x}{2} \sigma_3 + \frac{\theta_\infty}{2} \frac{G^{-1} \sigma_3 G}{v} + \frac{G^{-1} A_1 G}{v-1} \right] \tilde{\Psi}_{\text{OUT}}.$$

This form is that of a system of isomonodromy deformation for the fifth Painlevé equation.

$$\Omega_1 = \text{diagonal part of } D_1.$$

For any sector of angular width  $\frac{\pi}{2} + \epsilon$ ,  $\epsilon > 0$  sufficiently small, there exist a unique solution of (60) with asymptotic expansion:

$$Y_2(z) \sim \left[ I + \frac{K_1}{z} + \frac{K_2}{z^2} + \dots \right] \exp \left\{ \frac{x^2}{2} \Lambda z^2 + x \Lambda z \right\} z^{\Lambda_1}, \quad z \rightarrow \infty.$$

$$\Lambda_1 = \text{diagonal part of } E_1.$$

We can always find two solutions  $Y_1(z)$  and  $Y_2(z)$  as above, such that the sectors where the asymptotic expansions hold are overlapping. We refer the reader to [3] for the general description of irregular system with a Stokes phenomenon and to appendix B for the computation of the matrices  $G_i, K_i, i = 1, 2, \dots$ .

Systems (57), (58) are *isomonodromic*. This imposes that  $\Omega_1$  and  $\Lambda_1$  must be independent of  $x$ . They are

$$\Omega_1 = \text{diagonal of } (-G^{-1}(A_0 + A_x)G),$$

$$\Lambda_1 = \text{diagonal of } (A_0 + A_x) = \begin{pmatrix} a(x) & 0 \\ 0 & -a(x) \end{pmatrix}.$$

We compute

$$G^{-1}(A_0 + A_x)G = \frac{1}{\theta_x} \begin{pmatrix} -(2s + \theta_x)a - s(s + \theta_x)b + c & (-2a - sb + \frac{c}{s})r \\ \frac{s}{r}(2(s + \theta_x)a + (s + \theta_x)^2b - c) & (2s + \theta_x)a + s(s + \theta_x)b - c \end{pmatrix}.$$

Since  $a, b, c$  vanish, the condition of isomonodromicity implies that

$$\Omega_1 = 0, \quad \Lambda_1 = 0.$$

This means that *the leading terms of  $a(x), b(x), c(x)$  satisfy the conditions*

$$a(x) = 0, \quad c(x) = s(s + \theta_x)b(x).$$

The above conditions mean that if  $b(x), c(x) = O(x^{\sigma_0})$ , then  $a(x)$  is of higher order, i.e. it vanishes faster than  $x^{\sigma_0}$ . Note that with this choice of  $a, b, c$  we get

$$G^{-1}(A_0 + A_x)G = \begin{pmatrix} 0 & br \\ \frac{s(s+\theta_x)}{r}b & 0 \end{pmatrix}.$$

We are ready to write the behaviour of  $\Psi_{\text{OUT}}$ :

$$\Psi_{\text{OUT}} = G \left[ I + \frac{G_1}{v} + \frac{G_2}{v^2} + \dots \right] \exp \left\{ x \frac{\theta_x}{2} \sigma_3 v \right\}, \quad v \rightarrow \infty$$

$$= G [I + G_1 \lambda + G_2 \lambda^2 + \dots] \exp \left\{ \frac{\theta_x}{2} \sigma_3 \frac{x}{\lambda} \right\}, \quad \lambda \rightarrow 0.$$

We use the formulae of appendix B to determine  $G_1$ :

$$(G_1)_{ij} = 2 \frac{[G^{-1}(A_0 + A_x)G]_{ij}}{x \theta_x ((\sigma_3)_{ii} - (\sigma_3)_{jj})}, \quad i \neq j.$$

$$(G_1)_{ii} = \frac{\theta_\infty}{2} (G^{-1} \sigma_3 G)_{ij} + 2 \frac{[G^{-1}(A_0 + A_x)G]_{ij} [G^{-1}(A_0 + A_x)G]_{ji}}{x \theta_x ((\sigma_3)_{jj} - (\sigma_3)_{ii})}.$$

In the second term of the last formula,  $j = 2$  if  $i = 1$ ,  $j = 1$  if  $i = 2$ . We compute

$$G^{-1} \sigma_3 G = \frac{1}{\theta_x} \begin{pmatrix} -(2s + \theta_x) & -2r \\ \frac{2s(s+\theta_x)}{r} & 2s + \theta_x \end{pmatrix}. \tag{61}$$

Therefore,

$$\begin{aligned} (G_1)_{12} &= \frac{r}{x\theta_x}b, & (G_1)_{21} &= -\frac{s(s+\theta_x)}{x\theta_x r}b, \\ (G_1)_{11} &= -\frac{\theta_\infty}{2\theta_x}(2s+\theta_x) + \frac{s(s+\theta_x)}{x\theta_x}b^2, & (G_1)_{22} &= -(G_1)_{11}. \end{aligned} \quad (62)$$

On the other hand, the local behaviour of  $\Psi_{\text{IN}}$  is

$$\begin{aligned} \Psi_{\text{IN}} &= \left[ I + \frac{K_1}{\mu} + \frac{K_2}{\mu^2} + \dots \right] \exp \left\{ x^2 \frac{\theta_\infty}{4} \sigma_3 \mu^2 + x \frac{\theta_\infty}{2} \sigma_3 \mu \right\}, & \mu &\rightarrow \infty \\ &= \left[ I + K_1 \frac{x}{\mu} + K_2 \frac{x^2}{\lambda^2} + \dots \right] \exp \left\{ \frac{\theta_\infty}{4} \sigma_3 \lambda^2 + \frac{\theta_\infty}{2} \sigma_3 \lambda \right\}, & \lambda &\rightarrow 0. \end{aligned}$$

We determine  $K_1$  from the formulae of appendix B:

$$K_1 = \text{diagonal part of } (-A) = \begin{pmatrix} -(s + \frac{\theta_x}{2}) & 0 \\ 0 & s + \frac{\theta_x}{2} \end{pmatrix}.$$

The matching condition,

$$\Psi_{\text{OUT}}(\lambda, x) \sim \Psi_{\text{IN}}(\lambda, x), \quad x \rightarrow 0, \quad |x|^{\delta_{\text{OUT}}} \leq |\lambda| \leq |x|^{\delta_{\text{IN}}},$$

is restricted to the overlapping sector where both expansions of  $\Psi_{\text{OUT}}$  and  $\Psi_{\text{IN}}$  hold. Noting that  $\Psi_{\text{OUT}} \sim G$  and  $\Psi_{\text{IN}} \sim I$ , we choose the new solution  $\Psi_{\text{OUT}} \mapsto \Psi_{\text{OUT}} G^{-1}$ . Then, we expand the exponents:

$$\Psi_{\text{OUT}} = [I + GG_1 G^{-1} \lambda + GG_2 G^{-1} \lambda^2 + \dots] \left[ I + \frac{\theta_x}{2} G \sigma_3 G^{-1} \frac{x}{\lambda} + \frac{\theta_x^2}{8} \frac{x^2}{\lambda^2} + \dots \right].$$

The point here is quite delicate. We consider the relation of dominance among terms—and write the leading terms of the expansion—as they are in case  $G_n(x)$ 's are not divergent when  $x \rightarrow 0$ . Keeping into account that  $\frac{\theta_x}{2} G \sigma_3 G^{-1} = -A$ , the dominant terms are

$$\Psi_{\text{OUT}}(\lambda, x) = I + GG_1 G^{-1} \lambda - A \frac{x}{\lambda} + O\left(\lambda^2, \frac{x^2}{\lambda^2}, x\right).$$

It is important to note that  $\lambda$  is dominant w.r.t.  $\frac{x}{\lambda}$ , because  $\delta_{\text{OUT}} < \frac{1}{2}$ ; namely,  $\lambda \sim x^\delta$  vanishes slower than  $\frac{x}{\lambda} \sim x^{1-\delta}$ , as  $x \rightarrow 0$ .

We expand the exponent in  $\Psi_{\text{IN}}$  and keep only the first dominant terms (in the spirit of the observation on the dominance relations made above):

$$\Psi_{\text{IN}}(\lambda, x) = I + \frac{\theta_\infty}{2} \sigma_3 \lambda + K_1 \frac{x}{\lambda} + O\left(\lambda^2, \frac{x^2}{\lambda^2}, x\right).$$

$\Psi_{\text{OUT}}$  and  $\Psi_{\text{IN}}$  match in the first term  $I$ . We impose the matching of the second term, namely the term in  $\lambda$ :

$$GG_1(x)G^{-1} \sim \frac{\theta_\infty}{2} \sigma_3, \quad x \rightarrow 0.$$

Namely,

$$G_1(x) \sim \frac{\theta_\infty}{2} G^{-1} \sigma_3 G, \quad x \rightarrow 0. \quad (63)$$

From the explicit form of  $G_1$  and  $G^{-1} \sigma_3 G$  given above, we conclude that the matching is satisfied if and only if

$$b(x) \sim -x\theta_\infty \quad \text{and} \quad \sigma_0 = 1.$$

The error in  $b(x)$  is of higher order w.r.t.  $x$ . The determination of the leading behaviour of  $A_0 + A_x$  is complete, because  $c(x) \sim s(s + \theta_x)b(x)$ ,  $x \rightarrow 0$ . Namely,

$$c(x) \sim -x\{s(s + \theta_x)\theta_\infty\}, \quad a(x) = o(x).$$

With such a choice of  $b(x)$ , one can verify that the terms in  $\Psi_{\text{OUT}}$  and  $\Psi_{\text{IN}}$  which follow the second (i.e., which follow the term in  $\lambda$ ) are actually of higher order in  $x$ . Nevertheless,  $\Psi_{\text{OUT}}$  and  $\Psi_{\text{IN}}$  match only in the first and second terms, being already the off-diagonal entries of the third term not matching (i.e.,  $-A$  and  $K_1 = \text{diagonal part of } -A$  respectively.)

5.4. Matching for the second choice:  $\frac{1}{2} < \delta_{\text{IN}} \leq \delta_{\text{OUT}} < \frac{2}{3}$

We rewrite the systems in the convenient form:

$$\begin{aligned} \nu &:= \frac{1}{\lambda}, & \mu &:= \frac{\lambda}{x}; & \frac{d\Psi_{\text{OUT}}}{d\nu} &= \left[ -x^2 A \nu - x A - \frac{A_0 + A_x}{\nu} - \frac{\theta_\infty}{2} \frac{\sigma_3}{\nu(\nu - 1)} \right] \Psi_{\text{OUT}}; \\ \frac{d\Psi_{\text{IN}}}{d\mu} &= \left[ x \frac{\theta_\infty}{2} \sigma_3 + \frac{A_0}{\mu} + \frac{A_x}{\mu - 1} \right] \Psi_{\text{IN}} = \left[ x \frac{\theta_\infty}{2} \sigma_3 + \frac{A_0 + A_x}{\mu} - \frac{A_0}{\mu(\mu - 1)} \right] \Psi_{\text{IN}}. \end{aligned} \quad (64)$$

Then, substitute  $A_0 \mapsto -A$  in the last term.

We rewrite the systems at infinity<sup>6</sup>:

$$\begin{aligned} \frac{d\tilde{\Psi}_{\text{OUT}}}{d\nu} &= \left[ x^2 \frac{\theta_x}{2} \sigma_3 \nu + x \frac{\theta_x}{2} \sigma_3 - \frac{G^{-1}(A_0 + A_x)G}{\nu} - \frac{\theta_\infty}{2} G^{-1} \sigma_3 G \left( \frac{1}{\nu^2} + \frac{1}{\nu^3} + \dots \right) \right] \tilde{\Psi}_{\text{OUT}}, \\ \frac{d\Psi_{\text{IN}}}{d\mu} &= \left[ x \frac{\theta_\infty}{2} \sigma_3 + \frac{A_0 + A_x}{\mu} + A \left( \frac{1}{\mu^2} + \frac{1}{\mu^3} + \dots \right) \right] \Psi_{\text{IN}}, \quad \nu, \mu \rightarrow \infty. \end{aligned} \quad (65)$$

This time the system of  $\tilde{\Psi}_{\text{OUT}}$  is in the form (60), while the system of  $\Psi_{\text{IN}}$  is in the form (59), where

$$\begin{aligned} \Omega &= x \frac{\theta_\infty}{2} \sigma_3, & \Omega_1 &= \text{diagonal part of } (A_0 + A_x), \\ \Lambda &= \frac{\theta_x}{2} \sigma_3, & \Lambda_1 &= \text{diagonal part of } (-G^{-1}(A_0 + A_x)G). \end{aligned}$$

We impose that  $\Omega_1$  and  $\Lambda_1$  do not depend on  $x$ , and we get the conditions  $a = 0$ ,  $c = s(s + \theta_x)b$ . Then, we choose the following solutions:

$$\begin{aligned} \Psi_{\text{OUT}} &= G \left[ I + \frac{K_1}{\nu} + \dots \right] \exp \left\{ x^2 \frac{\theta_x}{4} \sigma_3 \nu^2 + x \frac{\theta_x}{2} \sigma_3 \nu \right\} G^{-1} \\ &= I + \frac{\theta_x}{2} G \sigma_3 G^{-1} \frac{x}{\lambda} + G K_1 G^{-1} \lambda + O \left( \frac{x^2}{\lambda^2}, x, \lambda^2 \right), \quad \nu \rightarrow \infty. \\ \Psi_{\text{IN}} &= \left[ I + \frac{G_1}{\mu} + \dots \right] \exp \left\{ x \frac{\theta_\infty}{2} \sigma_3 \mu \right\} = I + G_1 \frac{x}{\lambda} + \frac{\theta_\infty}{2} \sigma_3 \lambda + O \left( \frac{x^2}{\lambda^2}, x, \lambda^2 \right), \quad \mu \rightarrow \infty. \end{aligned}$$

The relation of dominance among terms are considered as if  $G_n$ 's and  $K_n$ 's do not diverge as  $x \rightarrow 0$ . The matching condition,

$$\Psi_{\text{OUT}}(\lambda, x) \sim \Psi_{\text{IN}}(\lambda, x), \quad x \rightarrow 0, \quad |x|^{\delta_{\text{OUT}}} \leq |\lambda| \leq |x|^{\delta_{\text{IN}}},$$

is restricted to the overlapping sector where both expansions of  $\Psi_{\text{OUT}}$  and  $\Psi_{\text{IN}}$  hold. We note that  $\frac{x}{\lambda}$  vanishes slower than  $\lambda$ , because  $\delta_{\text{IN}} > \frac{1}{2}$  (namely,  $x/\lambda \sim x^{1-\delta}$ ,  $\lambda \sim x^\delta$ ,  $\delta > 1/2$ ).

<sup>6</sup> System (64) is in the form of a system of isomonodromy deformation for the fifth Painlevé equation.

$\Psi_{\text{IN}}$  and  $\Psi_{\text{OUT}}$  automatically match in the first term  $I$ . We impose the matching of the second leading term, i.e. the term in  $\frac{x}{\lambda}$ :

$$G_1(x) \sim \frac{\theta_x}{2} G \sigma_3 G^{-1} \equiv -A, \quad x \rightarrow 0. \quad (66)$$

As for  $G_1$ , the formulae in appendix B give

$$(G_1)_{ij} = -2 \frac{(A_0 + A_x)_{ij}}{x \theta_\infty [(\sigma_3)_{ii} - (\sigma_3)_{jj}]}, \quad i \neq j.$$

$$(G_1)_{ii} = -(A)_{ii} + 2 \frac{(A_0 + A_x)_{ij} (A_0 + A_x)_{ji}}{x \theta_\infty [(\sigma_3)_{jj} - (\sigma_3)_{ii}]}.$$

In the last formula,  $j = 2$  if  $i = 1$ ,  $j = 1$  if  $i = 2$ . Explicitly,

$$(G_1)_{12} = -\frac{r}{x \theta_\infty} b, \quad (G_1)_{21} = \frac{s(s + \theta_x)}{x \theta_\infty r} b,$$

$$(G_1)_{11} = -(A)_{11} - \frac{s(s + \theta_x)}{x \theta_\infty} b^2, \quad (G_1)_{22} = -(A)_{22} + \frac{s(s + \theta_x)}{x \theta_\infty} b^2. \quad (67)$$

Therefore, (66)  $\iff b(x) \sim -x \theta_\infty$  and  $\sigma_0 = 1$ . As it must be, we get the same result as the matching for the first choice.

5.5. *Critical matching:*  $\frac{1}{2} - \epsilon < \delta_{\text{IN}} \leq \delta_{\text{OUT}} < \frac{1}{2} + \epsilon$

In between the first and the second choices—which hold, respectively, for  $\frac{1}{3} < \delta_{\text{IN}} \leq \delta_{\text{OUT}} < \frac{1}{2}$  and  $\frac{1}{2} < \delta_{\text{IN}} \leq \delta_{\text{OUT}} < \frac{2}{3}$ —we can also consider the following approximations of system (1):

$$\frac{d\Psi_{\text{OUT}}}{d\lambda} = \left[ \frac{x A}{\lambda^2} + \frac{A_0 + A_x}{\lambda} - \frac{\theta_\infty}{2} \frac{\sigma_3}{\lambda - 1} \right] \Psi_{\text{OUT}},$$

$$\frac{d\Psi_{\text{IN}}}{d\lambda} = \left[ \frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} + \frac{\theta_\infty}{2} \sigma_3 \right] \Psi_{\text{IN}}.$$

Rigorously speaking, the two systems cannot be considered simultaneously when  $\sigma_0 = 1$ . But we can consider  $\sigma_0 = 1$  as a ‘limit’ value—or ‘critical’ value—for the matching of the two above systems in the region specified by  $\frac{1}{2} - \epsilon < \delta_{\text{IN}} \leq \delta_{\text{OUT}} < \frac{1}{2} + \epsilon$ , where  $\epsilon > 0$  is sufficiently small. We write again  $\Psi_{\text{OUT}} =: G \tilde{\Psi}_{\text{OUT}}$ . Then,

$$\frac{d\tilde{\Psi}_{\text{OUT}}}{d\nu} = \left[ x \frac{\theta_x}{2} \sigma_3 - \frac{G^{-1}(A_0 + A_x)G}{\nu} - \frac{\theta_\infty}{2} G^{-1} \sigma_3 G \left( \frac{1}{\nu^2} + \frac{1}{\nu^3} + \dots \right) \right] \tilde{\Psi}_{\text{OUT}}, \quad \nu \rightarrow \infty;$$

$$\frac{d\Psi_{\text{IN}}}{d\mu} = \left[ x \frac{\theta_\infty}{2} \sigma_3 + \frac{A_0 + A_x}{\mu} + A \left( \frac{1}{\mu^2} + \frac{1}{\mu^3} + \dots \right) \right] \Psi_{\text{IN}}, \quad \mu \rightarrow \infty.$$

When we impose isomonodromicity conditions, the diagonal parts of  $A_0 + A_x$  and  $G^{-1}(A_0 + A_x)G$  must be independent of  $x$ . This gives again  $a = 0$ ,  $c = s(s + \theta_x)b$ . Then, we choose the fundamental solutions:

$$\Psi_{\text{OUT}} = G \left[ I + \frac{G_1^{\text{OUT}}}{\nu} + \dots \right] \exp \left\{ x \frac{\theta_x}{2} \sigma_3 \nu \right\} G^{-1}$$

$$= I + G G_1^{\text{OUT}} G^{-1} \lambda + \frac{\theta_x}{2} G \sigma_3 G^{-1} \frac{x}{\lambda} + O \left( \lambda^2, x, \frac{x^2}{\lambda^2} \right)$$

$$\Psi_{\text{IN}} = \left[ I + G_1^{\text{IN}} \frac{x}{\lambda} + \dots \right] \exp \left\{ x \frac{\theta_\infty}{2} \mu \right\} = I + \frac{\theta_\infty}{2} \sigma_3 \lambda + G_1^{\text{IN}} \frac{x}{\lambda} + O \left( \lambda^2, x, \frac{x^2}{\lambda^2} \right).$$

We match them for  $\lambda \sim x^\delta$  and  $\delta \in (1/2 - \epsilon, 1/2 + \epsilon)$ , in the overlapping sector where the above expansions hold. Here, both  $\lambda \sim x^\delta$  and  $x/\lambda \sim x^{1-\delta}$  are the dominant terms. The matching conditions are  $GG_1^{\text{OUT}}G^{-1} \sim \frac{\theta_\infty}{2}\sigma_3$  and  $G_1^{\text{IN}} \sim \frac{\theta_x}{2}G\sigma_3G^{-1}$ . Namely,

$$G_1^{\text{OUT}}(x) \sim \frac{\theta_\infty}{2}G^{-1}\sigma_3G, \quad G_1^{\text{IN}}(x) \sim -A, \quad x \rightarrow 0. \tag{68}$$

The matrix  $\frac{\theta_\infty}{2}G^{-1}\sigma_3G$  can be derived from (61). The matrix  $G_1^{\text{OUT}}$  is (62), the matrix  $G_1^{\text{IN}}$  is (67). Condition (68) is inclusive of both (63) and (66). Therefore, (68)  $\iff b(x) \sim -x\theta_\infty, x \rightarrow 0$ . This is again the expected result.

5.6. Higher order terms

The final result obtained above is

$$A_0 + A_x = \begin{pmatrix} 0 & -r\theta_\infty x \\ -\frac{(s+\theta_x)s\theta_\infty}{r}x & 0 \end{pmatrix} + o(x), \quad A_1 = -\frac{\theta_\infty}{2}\sigma_3 - (A_0 + A_x), \tag{69}$$

$$A_x = \begin{pmatrix} s + \frac{\theta_x}{2} & -r \\ \frac{(s+\theta_x)s}{r} & -s - \frac{\theta_0}{2} \end{pmatrix} + o(1), \quad A_0 = -\begin{pmatrix} s + \frac{\theta_x}{2} & -r \\ \frac{(s+\theta_x)s}{r} & -s - \frac{\theta_0}{2} \end{pmatrix} + o(1). \tag{70}$$

Let us substitute the above results into (3). We obtain the first term with no error:

$$y(x) \sim \frac{1}{1 - \theta_\infty}, \quad x \rightarrow 0.$$

Here,  $r$  and  $s$  do not appear. Nevertheless, if we substitute in (PVI) the series  $y = \frac{1}{1-\theta_\infty} + \sum_{n=1}^\infty b_n x^n$ , we can compute recursively all the terms, for  $\theta_0 = \pm\theta_x$  and  $\theta_1 = \pm\theta_\infty$ . We find a series

$$y(x) = \frac{1}{1 - \theta_\infty} + ax + \sum_{n=0}^\infty b_n(a; \theta_\infty, \theta_0)x^n, \quad x \rightarrow 0, \tag{71}$$

where  $a$  is an arbitrary parameter. This parameter is actually a function of  $s$ , as we prove now. The convergence of the Taylor expansion can be proved by a Briot–Bouquet-like argument. This will not be done here. The reader can find a similar proof in [18] and the general procedure in [14].

5.6.1. Determination of  $a = a(s)$ . System (1) is isomonodromic. This determines the structure  $A_x, A_0A_1$  as can be found in [17], appendix C, formulae (C.47), (C.49), (C.51), (C.52), (C.55). If we substitute (71) in the formulae, we get a Taylor expansion for the matrix elements, in terms of the parameter  $a$ . The leading terms have exactly the structure of (69) and (70). We can identify the leading terms to express  $a$  as a function of  $s$  and  $r$ . The computations are quite long, so we give the result. When we write the leading terms as a function of  $a$  and impose that they coincide with (69) and (70), we find

$$a = \frac{\theta_\infty(2s + \theta_x + 1)}{2(\theta_\infty - 1)} \in \mathbf{C}.$$

The higher order terms are Taylor expansions. Explicitly, the first terms are

$$\begin{aligned} A_1 &= -\frac{\theta_\infty}{2}\sigma_3 - (A_0 + A_x) \\ &= \begin{pmatrix} -\frac{\theta_\infty}{2} + (s + \theta_x)s\theta_\infty x^2 & r\theta_\infty \left\{ x - \frac{(\theta_\infty+1)(2s+\theta_x-1)}{2}x^2 \right\} \\ \frac{s\theta_\infty}{r} \left\{ (\theta_x + s)x - \frac{(\theta_\infty-1)(\theta_x+s)(2s+\theta_x+1)s}{2}x^2 \right\} & \frac{\theta_\infty}{2} - (\theta_x + s)s\theta_\infty x^2 \end{pmatrix} + O(x^3), \end{aligned} \tag{72}$$



$$A_x = \begin{pmatrix} (s + \frac{\theta_x}{2}) - 2(s + \theta_x)s\theta_\infty x & -r \{1 - (2s + \theta_x - 1)\theta_\infty x\} \\ \frac{s(s+\theta_x)}{r} \{1 - (2s + \theta_x + 1)\theta_\infty x\} & -(s + \frac{\theta_x}{2}) + 2(s + \theta_x)s\theta_\infty x \end{pmatrix} + O(x^2), \quad (73)$$

$$A_0 = \begin{pmatrix} -(s + \frac{\theta_x}{2}) + 2(s + \theta_x)s\theta_\infty x & r \{1 - (2s + \theta_x)\theta_\infty x\} \\ -\frac{s(s+\theta_x)}{r} \{1 - (2s + \theta_x)\theta_\infty x\} & (s + \frac{\theta_x}{2}) - 2(s + \theta_x)s\theta_\infty x \end{pmatrix} + O(x^2). \quad (74)$$

The above expansions are enough to obtain first two leading terms of (3):

$$y(x) = \frac{1}{1 - \theta_\infty} + \frac{\theta_\infty(2s + \theta_x + 1)}{2(\theta_\infty - 1)}x + O(x^2).$$

Note that  $r$  simplifies. This is solution (16).

### 5.7. Monodromy data

We assume that the matching has been completed as above, and in particular  $\sigma_0 = 1$ . Thus, system (1) can be approximated by

$$\frac{d\Psi_{\text{OUT}}}{d\lambda} = \left[ \frac{A_0 + A_x}{\lambda} + \frac{x A_x}{\lambda^2} - \frac{\theta_\infty}{2} \frac{\sigma_3}{\lambda - 1} \right] \Psi_{\text{OUT}}, \quad \text{for } |\lambda| \geq |x|^\delta, \quad \delta < \frac{1}{2}; \quad (75)$$

or

$$\frac{d\Psi_{\text{IN}}}{d\lambda} = \left[ \frac{A_0}{\lambda} + \frac{A_x}{\lambda} + \frac{\theta_\infty}{2} \sigma_3 \right] \Psi_{\text{IN}}, \quad \text{for } |\lambda| \leq |x|^\delta, \quad \delta > \frac{1}{2}. \quad (76)$$

The first two leading terms are

$$\Psi_{\text{OUT}} = I + \frac{\theta_\infty}{2} \sigma_3 \lambda + O\left(\frac{x}{\lambda}\right), \quad \lambda \sim x^\delta \rightarrow 0, \quad \delta < \frac{1}{2}; \quad (77)$$

$$\Psi_{\text{IN}} = I - A \frac{x}{\lambda} + O(\lambda), \quad \lambda \sim x^\delta \rightarrow 0, \quad \delta > \frac{1}{2}. \quad (78)$$

The above solutions match in the first three terms in the ‘critical’ region  $\lambda \sim x^\delta, \delta \simeq \frac{1}{2}$  (the region is restricted to a sector). Namely,

$$\Psi_{\text{OUT}} \sim \Psi_{\text{IN}} \sim I + \frac{\theta_\infty}{2} \sigma_3 \lambda - A \frac{x}{\lambda} + O\left(x, \frac{x^2}{\lambda^2}, \lambda^2\right), \quad \lambda \sim x^\delta, \quad \delta \simeq \frac{1}{2}. \quad (79)$$

Now, for  $\delta < \frac{1}{2}$ , we have

$$\frac{A_0 + A_x}{\lambda} \sim x^{1-\delta} \rightarrow 0, \quad \frac{x A_x}{\lambda^2} \sim x^{1-2\delta} \rightarrow 0.$$

Thus, (75) can be further reduced to

$$\frac{d\hat{\Psi}_{\text{OUT}}}{d\lambda} = -\frac{\theta_\infty}{2} \frac{\sigma_3}{\lambda - 1} \hat{\Psi}_{\text{OUT}}. \quad (80)$$

In system (76) we rewrite

$$\frac{A_0}{\lambda} + \frac{A_x}{\lambda - x} = \frac{A_0 + A_x}{\lambda - x} - \frac{x A_0}{\lambda(\lambda - x)}.$$

Then, we observe that, for  $\delta > \frac{1}{2}$ , we have the behaviours

$$\frac{A_0 + A_x}{\lambda - x} \sim x^{1-\delta} \rightarrow 0, \quad \frac{x A_0}{\lambda(\lambda - x)} \sim -\frac{x A}{\lambda(\lambda - x)} \sim x^{1-2\delta} \rightarrow \infty, \quad \delta > \frac{1}{2}.$$

Thus, as  $x \rightarrow 0$ , system (76) can be further reduced to

$$\frac{d\hat{\Psi}_{\text{IN}}}{d\lambda} = \frac{x A}{\lambda(\lambda - x)} \hat{\Psi}_{\text{IN}}. \tag{81}$$

System (80) has the same monodromy of (1) at  $\lambda = 1, \infty$ . System (81) has the same monodromy of (1) at  $\lambda = 0, x$ .

*Matching  $\Psi \leftrightarrow \hat{\Psi}_{\text{OUT}}$*

We choose  $\hat{\Psi}_{\text{OUT}}$  such that it matches with  $\Psi$  at  $\lambda = \infty$ . The behaviour  $\Psi$  is

$$\Psi(\lambda, x) = \left[ I + O\left(\frac{1}{\lambda}\right) \right] \lambda^{-\frac{\theta_\infty}{2}\sigma_3}, \quad \lambda \rightarrow \infty,$$

for  $\theta_\infty \notin \mathbf{Z}$ . The solution of (80) with the same behaviour is

$$\hat{\Psi}_{\text{OUT}}(\lambda) := (\lambda - 1)^{-\frac{\theta_\infty}{2}\sigma_3}. \tag{82}$$

As a consequence, the monodromy of  $\Psi$  at  $\lambda = 1, \infty$  coincides with that of  $\hat{\Psi}_{\text{OUT}}$ . To compute it, we consider the loops  $\lambda - 1 \mapsto (\lambda - 1) e^{2\pi i}$  and  $\lambda \mapsto \lambda e^{2\pi i}$ . The corresponding monodromy is

$$M_1 = M_\infty = \exp\{i\pi\theta_\infty\sigma_3\}.$$

*Matching  $\hat{\Psi}_{\text{OUT}} \leftrightarrow \Psi_{\text{OUT}}$*

We match  $\hat{\Psi}_{\text{OUT}}$  with  $\Psi_{\text{OUT}}$  for  $x/\lambda \rightarrow 0$ . Let us choose the branch  $(\lambda - 1) = (1 - \lambda) e^{i\pi}$ ,  $(1 - \lambda) > 0$  for  $0 < \lambda < 1$ . Solution (82) has expansion

$$\hat{\Psi}_{\text{OUT}}(\lambda) = e^{-i\frac{\pi}{2}\theta_\infty\sigma_3} \left[ I + \frac{\theta_\infty}{2}\sigma_3\lambda + O(\lambda^2) \right], \quad \lambda \rightarrow 0.$$

Therefore, for  $\lambda \rightarrow 0$ ,  $\hat{\Psi}_{\text{OUT}}$  matches with  $\Psi_{\text{OUT}} e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}$ , where  $\Psi_{\text{OUT}}$  is (77).

*Matching  $\Psi_{\text{OUT}} \leftrightarrow \Psi_{\text{IN}}$* . This is (79).

*Matching  $\Psi_{\text{IN}} \leftrightarrow \Psi$* . The above matchings imply that  $\Psi_{\text{IN}} e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}$  matches with  $\Psi$  (where  $\Psi_{\text{IN}}$  is (78)).

*Matching  $\Psi_{\text{IN}} \leftrightarrow \hat{\Psi}_{\text{IN}}$*

In order to determine the monodromy of system (1) at  $\lambda = 0, x$ , we need to find a fundamental solution  $\hat{\Psi}_{\text{IN}}$  of (81), that matches with  $\Psi_{\text{IN}} e^{-i\frac{\pi}{2}\sigma_3}$  for  $x/\lambda \rightarrow 0$ , where  $\Psi_{\text{IN}}$  is (78). A fundamental solution satisfying these requirements is

$$\hat{\Psi}_{\text{IN}}(\lambda, x) := G \left[ \left(1 - \frac{x}{\lambda}\right)^{-\frac{\theta_x}{2}\sigma_3} \right] G^{-1} e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}.$$

Actually, this has the behaviour

$$\hat{\Psi}_{\text{IN}}(\lambda, x) = \left[ I + \frac{\theta_x}{2} G \sigma_3 G^{-1} \frac{x}{\lambda} + O\left(\frac{x^2}{\lambda^2}\right) \right] e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}, \quad \frac{x}{\lambda} \rightarrow 0,$$

where  $\frac{\theta_x}{2} G \sigma_3 G^{-1} \equiv -A$ . The first two terms match with  $\Psi_{\text{IN}} e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}$ , as required.

As a consequence of the matching, the monodromy matrices of  $\Psi$  and  $\hat{\Psi}_{\text{IN}}$  at  $\lambda = 0, x$  coincide. To compute them, we write the local behaviours (for  $x \neq 0$  fixed) of  $\hat{\Psi}_{\text{IN}}$ :

$$\hat{\Psi}_{\text{IN}}(\lambda, x) = \begin{cases} [G x^{\frac{\theta_x}{2}\sigma_3}][1 + O(\lambda - x)](\lambda - x)^{-\frac{\theta_x}{2}\sigma_3} [G^{-1} e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}], & \lambda \rightarrow x, \\ [G(-x)^{-\frac{\theta_x}{2}\sigma_3}][1 + O(\lambda)]\lambda^{\frac{\theta_x}{2}\sigma_3} [G^{-1} e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}], & \lambda \rightarrow 0. \end{cases}$$

It is not necessary to specify the branch of  $(\pm x)^{-\frac{\theta_x}{2}\sigma_3}$ . We restrict to the case  $\theta_0$  and  $\theta_x \notin \mathbf{Z}$ , so that the matrices  $R_0, R_x$  in (14) are zero and the matching of  $\Psi$  with the above behaviours of  $\hat{\Psi}_{\text{IN}}$  is realized.

We can compute the monodromy matrices for  $\lambda \mapsto \lambda e^{2\pi i}$  and  $(\lambda - x) \mapsto (\lambda - x) e^{2\pi i}$ , respectively:

$$M_0 = e^{i\frac{\pi}{2}\theta_\infty\sigma_3} G \exp\{i\pi\theta_x\sigma_3\} G^{-1} e^{-i\frac{\pi}{2}\theta_\infty\sigma_3},$$

$$M_x = e^{i\frac{\pi}{2}\theta_\infty\sigma_3} G \exp\{-i\pi\theta_x\sigma_3\} G^{-1} e^{-i\frac{\pi}{2}\theta_\infty\sigma_3}.$$

Note that  $M_1 = M_\infty$  is invariant for  $M_1 \mapsto e^{-i\frac{\pi}{2}\theta_\infty\sigma_3} M_1 e^{i\frac{\pi}{2}\theta_\infty\sigma_3}$ . With this fact in mind, we obtain the result of theorem 3, point (b). In particular, computing the trace of  $M_0 M_1$ , we get

$$s = \frac{\theta_x [2 \cos(\pi(\theta_\infty + \theta_x)) - \text{tr}(M_1 M_0)]}{2[\cos(\pi(\theta_\infty - \theta_x)) - \cos(\pi(\theta_\infty + \theta_x))]}.$$

**6. Case  $\sigma = \pm(\theta_1 - \theta_\infty), \pm(\theta_1 + \theta_\infty)$ : solution (15)**

This case shows new features, namely  $r$  and  $r_1$  may be functions of  $x$ . For  $\sigma = \pm(\theta_1 - \theta_\infty), \pm(\theta_1 + \theta_\infty)$  the matrices (38), (39) become

$$\sigma = \pm(\theta_1 - \theta_\infty): \quad \hat{A}_1 = \begin{pmatrix} -\frac{\theta_1}{2} & -r_1 \\ 0 & \frac{\theta_1}{2} \end{pmatrix}, \quad \hat{A}_0 + \hat{A}_x = \begin{pmatrix} \frac{\theta_1 - \theta_\infty}{2} & r_1 \\ 0 & -\frac{\theta_1 - \theta_\infty}{2} \end{pmatrix},$$

$$\sigma = \pm(\theta_1 + \theta_\infty): \quad \hat{A}_1 = \begin{pmatrix} \frac{\theta_1}{2} & -r_1 \\ 0 & -\frac{\theta_1}{2} \end{pmatrix}, \quad \hat{A}_0 + \hat{A}_x = \begin{pmatrix} -\frac{\theta_1 + \theta_\infty}{2} & r_1 \\ 0 & \frac{\theta_1 + \theta_\infty}{2} \end{pmatrix}. \tag{83}$$

The transpose matrices may be considered (namely, redefine  $r_1 \mapsto -\frac{[\sigma^2 - (\theta_1 - \theta_\infty)^2][\sigma^2 - (\theta_1 + \theta_\infty)^2]}{16\theta_\infty^2 r_1}$ ).

The matrices  $\hat{A}_0$  and  $\hat{A}_x$  are again (44) and (45). For definiteness, we will consider the upper triangular matrices  $\hat{A}_1$  and  $\hat{A}_0 + \hat{A}_x$ , and the choice  $\sigma = \theta_1 - \theta_\infty$ . We distinguish three cases:

- (I)  $r_1$  is a non-zero constant. In this case, we just repeat the general matching procedure and find  $y(x)$  as in (48).  $r \neq 0$  is constant.
- (II)  $r_1 = 0$  and  $r$  constant. This is a very easy case, because  $\hat{A}_1 = -\frac{\theta_1}{2}\sigma_3, \hat{A}_0 + \hat{A}_x = \frac{\theta_1 - \theta_\infty}{2}\sigma_3$ . Therefore, a fundamental solution of (37) is

$$\Psi_{\text{OUT}}(\lambda) = (\lambda - 1)^{-\frac{\theta_1}{2}\sigma_3} \lambda^{\frac{\theta_1 - \theta_\infty}{2}\sigma_3} (-1)^{-\frac{\theta_1}{2}\sigma_3} = \left[ I + \frac{\theta_1}{2}\sigma_3 + O(\lambda^2) \right] \lambda^{\frac{\theta_1 - \theta_\infty}{2}\sigma_3}, \quad \lambda \rightarrow 0.$$

The solution of (43), with  $r$  constant, is the same as the general case. The matching is possible as in the general case, with  $K_0 = \begin{pmatrix} x^{\frac{\theta_1 - \theta_\infty}{2}} & 0 \\ 0 & x^{\frac{\theta_\infty - \theta_1}{2}} \end{pmatrix}$ . Therefore,

$$\hat{A}_0 = \begin{pmatrix} \frac{\theta_0^2 - \theta_x^2 + \sigma^2}{4\sigma} & r x^{\theta_1 - \theta_\infty} \\ -\frac{[\sigma^2 - (\theta_0 - \theta_x)^2][\sigma^2 - (\theta_0 + \theta_x)^2]}{16\sigma^2 r} x^{\theta_\infty - \theta_1} & -\frac{\theta_0^2 - \theta_x^2 + \sigma^2}{4\sigma} \end{pmatrix}.$$

We obtain

$$y(x) \sim \frac{x(\hat{A}_0)_{12}}{x[(\hat{A}_0)_{12} + (\hat{A}_1)_{12}] - (\hat{A}_1)_{12}} = \frac{r x^{\theta_1 - \theta_\infty}}{x[x^{\theta_1 - \theta_\infty} + 0] - 0} = 1.$$

This is the *singular solution*  $y(x) \equiv 1$ .

The case  $r_1 = r_1(x) \rightarrow 0$  may give a non-singular  $y(x)$ , provided that also  $r = r(x) \rightarrow 0$ . This will be proved below.

(III) Case  $r_1 = r_1(x)$ . *A priori*, we do not know if it is possible to have an  $x$ -dependent  $r_1$ . This is not actually possible if we approximate (1) with the Fuchsian system (37) (i.e., (10)). Actually, (37) is a system with reducible monodromy (upper triangular):

$$\frac{d\Psi_{\text{OUT}}}{d\lambda} = \left\{ \begin{pmatrix} \frac{\theta_1 - \theta_\infty}{2} & r_1 \\ 0 & -\frac{\theta_1 - \theta_\infty}{2} \end{pmatrix} \frac{1}{\lambda} + \begin{pmatrix} -\frac{\theta_1}{2} & -r_1 \\ 0 & \frac{\theta_1}{2} \end{pmatrix} \frac{1}{\lambda - 1} \right\} \Psi_{\text{OUT}}.$$

There is a fundamental solution (obtained by variation of parameters):

$$\Psi_{\text{OUT}} = \begin{bmatrix} (1 - \lambda)^{-\frac{\theta_1}{2}} & 0 \\ 0 & (1 - \lambda)^{\frac{\theta_1}{2}} \end{bmatrix} \begin{bmatrix} 1 & \frac{r_1 F(1 - \theta_1, \theta_\infty - \theta_1, \theta_\infty - \theta_1 + 1; \lambda)}{\theta_\infty - \theta_1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \lambda^{\frac{\theta_1 - \theta_\infty}{2}} & 0 \\ 0 & \lambda^{-\frac{\theta_1 - \theta_\infty}{2}} \end{bmatrix}.$$

Here,  $F(\dots)$  denotes a Gauss hyper-geometric function. This solution has a diagonal monodromy matrix at  $\lambda = 0$  and an upper triangular monodromy matrix at  $\lambda = 1$ , with  $r_1$  in the (1, 2) matrix element. Therefore,  $r_1$  must be independent of the monodromy-preserving deformation  $x$ . We are sent back to case (I) of constant  $r_1$ .

The only possibility for  $r_1$  to depend on  $x$  is that the matrices  $\hat{A}_0, \hat{A}_x$  have a behaviour, for  $x \rightarrow 0$ , such that system (1) is approximated by a system (8) with singularity of the second kind at  $\lambda = 0$ . Namely, system (1) must be approximated *at least* by

$$\frac{d\Psi_{\text{OUT}}}{d\lambda} = \left\{ \frac{x A_x}{\lambda^2} + \frac{A_0 + A_x}{\lambda} + \frac{A_1}{\lambda - 1} \right\} \Psi_{\text{OUT}}. \tag{84}$$

**Hypothesis.** We consider the case

$$x A_x = \begin{pmatrix} 0 & \rho(x) \\ 0 & 0 \end{pmatrix} + \text{higher orders}, \quad x \rightarrow 0.$$

In the above hypothesis, (84) is

$$\frac{d\Psi_{\text{OUT}}}{d\lambda} = \left\{ \begin{pmatrix} 0 & \rho(x) \\ 0 & 0 \end{pmatrix} \frac{1}{\lambda^2} + \begin{pmatrix} \frac{\theta_1 - \theta_\infty}{2} & r_1 \\ 0 & \frac{\theta_1 - \theta_\infty}{2} \end{pmatrix} \frac{1}{\lambda} + \begin{pmatrix} -\frac{\theta_1}{2} & -r_1 \\ 0 & \frac{\theta_1}{2} \end{pmatrix} \frac{1}{\lambda - 1} \right\} \Psi_{\text{OUT}}. \tag{85}$$

Let us write a  $\Psi_{\text{OUT}}$  as a vector  $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ . The system becomes

$$\begin{aligned} \frac{d\psi_1}{d\lambda} &= \left( \frac{\theta_1 - \theta_\infty}{2\lambda} - \frac{\theta_1}{2(\lambda - 1)} \right) \psi_1 + \left( \frac{r_1}{\lambda} + \frac{\rho}{\lambda^2} - \frac{r_1}{\lambda - 1} \right) \psi_2, \\ \frac{d\psi_2}{d\lambda} &= \left( \frac{\theta_\infty - \theta_1}{2\lambda} + \frac{\theta_1}{2(\lambda - 1)} \right) \psi_2. \end{aligned}$$

This system is solvable by variation of parameters. Let  $C_1$  and  $C_2$  be integration constants. The general solution is

$$\psi_2(\lambda) = C_2 \lambda^{\frac{\theta_\infty - \theta_1}{2}} (\lambda - 1)^{\frac{\theta_1}{2}}, \tag{86}$$

$$\begin{aligned} \psi_1(\lambda) &= C_1 \lambda^{\frac{\theta_1 - \theta_\infty}{2}} (\lambda - 1)^{-\frac{\theta_1}{2}} + C_2 e^{i\pi\theta_1} \left[ \frac{\rho}{\theta_\infty - \theta_1 - 1} \frac{F(1 - \theta_1, \theta_\infty - \theta_1 - 1, \theta_\infty - \theta_1; \lambda)}{\lambda} \right. \\ &\quad \left. + \frac{r_1 - \rho}{\theta_\infty - \theta_1} F(1 - \theta_1, \theta_\infty - \theta_1, \theta_\infty - \theta_1 + 1; \lambda) \right] \lambda^{\frac{\theta_\infty - \theta_1}{2}} (\lambda - 1)^{-\frac{\theta_1}{2}}. \end{aligned} \tag{87}$$

Here,  $F(\dots)$  denotes a Gauss hyper-geometric equation. The choice of the branch is such that for  $0 < \lambda < 1$ , we have  $0 < 1 - \lambda = e^{-i\pi}(\lambda - 1)$ .

In order to write the local behaviour for  $\lambda \rightarrow 0$ , we expand the hyper-geometric functions and  $(\lambda - 1)^{\pm\theta_1/2} = (e^{i\pi}(1 - \lambda))^{\pm\theta_1/2}$ :

$$\begin{aligned} \psi_1 = & C_1 e^{-i\frac{\pi}{2}\theta_1} \lambda^{\frac{\theta_1 - \theta_\infty}{2}} \left( 1 + \frac{\theta_1}{2} \lambda + O(\lambda^2) \right) + C_2 e^{i\frac{\pi}{2}\theta_1} \lambda^{\frac{\theta_\infty - \theta_1}{2}} \left[ -\frac{\rho}{1 + \theta_1 - \theta_\infty} \frac{1}{\lambda} \right. \\ & \left. + \frac{\rho\theta_1(\theta_1 - \theta_\infty + 2) - 2r_1(\theta_1 - \theta_\infty + 1)}{2(\theta_1 - \theta_\infty)(\theta_1 - \theta_\infty + 1)} + O(\lambda) \right]. \end{aligned}$$

$$\psi_2 = C_2 e^{i\frac{\pi}{2}\theta_1} \left( 1 - \frac{\theta_1}{2} \lambda + O(\lambda^2) \right) \lambda^{\frac{\theta_\infty - \theta_1}{2}}.$$

Therefore, we can take as a fundamental solution the following matrix (choose  $C_1 = e^{i\frac{\pi}{2}\theta_1}$ ,  $C_2 = e^{-i\frac{\pi}{2}\theta_1}$ ):

$$\begin{aligned} \Psi_{\text{OUT}} = & \begin{bmatrix} \left( 1 + \frac{\theta_1}{2} \lambda + \dots \right) \lambda^{\frac{\theta_1 - \theta_\infty}{2}} & \left[ -\frac{\rho}{1 + \theta_1 - \theta_\infty} \frac{1}{\lambda} + \frac{\rho\theta_1(\theta_1 - \theta_\infty + 2) - 2r_1(\theta_1 - \theta_\infty + 1)}{2(\theta_1 - \theta_\infty)(\theta_1 - \theta_\infty + 1)} + \dots \right] \lambda^{\frac{\theta_\infty - \theta_1}{2}} \\ 0 & \left( 1 - \frac{\theta_1}{2} \lambda + \dots \right) \lambda^{\frac{\theta_\infty - \theta_1}{2}} \end{bmatrix} \\ = & \left\{ \begin{pmatrix} 0 & -\frac{\rho}{1 + \theta_1 - \theta_\infty} \\ 0 & 0 \end{pmatrix} \frac{1}{\lambda} + \begin{pmatrix} 1 & \frac{\rho\theta_1(\theta_1 - \theta_\infty + 2) - 2r_1(\theta_1 - \theta_\infty + 1)}{2(\theta_1 - \theta_\infty)(\theta_1 - \theta_\infty + 1)} \\ 0 & 1 \end{pmatrix} + O(\lambda) \right\} \lambda^{\frac{\theta_1 - \theta_\infty}{2}} \sigma_3. \end{aligned}$$

### 6.1. Matching

The above solution must be matched with the solution of system (43), with  $\sigma = \theta_1 - \theta_\infty$ :

$$\Psi_0 \left( \frac{\lambda}{x} \right) = \left[ I + K_1 \frac{x}{\lambda} + O \left( \frac{x^2}{\lambda^2} \right) \right] \begin{pmatrix} \left( \frac{\lambda}{x} \right)^{\frac{\theta_1 - \theta_\infty}{2}} & 0 \\ 0 & \left( \frac{\lambda}{x} \right)^{\frac{\theta_\infty - \theta_1}{2}} \end{pmatrix}.$$

From a standard computation we find

$$K_1 = \begin{pmatrix} \frac{\theta_0^2 - (\theta_x - \sigma)^2}{4\sigma} & \frac{r}{\sigma + 1} \\ \frac{(\theta_0^2 - (\theta_x - \sigma)^2)(\theta_0^2 - (\theta_x + \sigma)^2)}{16\sigma^2(\sigma - 1)r} & \frac{(\theta_x + \sigma)^2 - \theta_0^2}{4\sigma(\sigma + 1)} \end{pmatrix}, \quad r \neq 0.$$

Note that  $r$  in (45) is constant, because the monodromy of (43) depends on  $r$ .

The matching relation  $\Psi_{\text{OUT}}(\lambda) \sim K_0(x)\Psi_0(\lambda/x)$  reads

$$\begin{aligned} & \left\{ \begin{pmatrix} 0 & -\frac{\rho}{1 + \theta_1 - \theta_\infty} \\ 0 & 0 \end{pmatrix} \frac{1}{\lambda} + \begin{pmatrix} 1 & \frac{\rho\theta_1(\theta_1 - \theta_\infty + 2) - 2r_1(\theta_1 - \theta_\infty + 1)}{2(\theta_1 - \theta_\infty)(\theta_1 - \theta_\infty + 1)} \\ 0 & 1 \end{pmatrix} + O(\lambda) \right\} \\ & \sim K_0(x) \left[ I + K_1 \frac{x}{\lambda} + \dots \right] \begin{pmatrix} x^{\frac{\theta_\infty - \theta_1}{2}} & 0 \\ 0 & x^{\frac{\theta_1 - \theta_\infty}{2}} \end{pmatrix}. \end{aligned}$$

Namely,

$$\begin{cases} \begin{pmatrix} 1 & \frac{\rho\theta_1(\theta_1 - \theta_\infty + 2) - 2r_1(\theta_1 - \theta_\infty + 1)}{2(\theta_1 - \theta_\infty)(\theta_1 - \theta_\infty + 1)} \\ 0 & 1 \end{pmatrix} \sim K_0(x) \begin{pmatrix} x^{\frac{\theta_\infty - \theta_1}{2}} & 0 \\ 0 & x^{\frac{\theta_1 - \theta_\infty}{2}} \end{pmatrix} \\ \begin{pmatrix} 0 & -\frac{\rho}{1 + \theta_1 - \theta_\infty} \\ 0 & 0 \end{pmatrix} \sim x K_0(x) F_1^{(0)} \begin{pmatrix} x^{\frac{\theta_\infty - \theta_1}{2}} & 0 \\ 0 & x^{\frac{\theta_1 - \theta_\infty}{2}} \end{pmatrix}. \end{cases}$$

The first equation above is

$$K_0(x) \sim \begin{pmatrix} 1 & \frac{\rho\theta_1(\theta_1 - \theta_\infty + 2) - 2r_1(\theta_1 - \theta_\infty + 1)}{2(\theta_1 - \theta_\infty)(\theta_1 - \theta_\infty + 1)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{\frac{\theta_1 - \theta_\infty}{2}} & 0 \\ 0 & x^{\frac{\theta_\infty - \theta_1}{2}} \end{pmatrix}.$$

We substitute this in the second equation. For simplicity, denote by  $K_{ij}$  the matrix elements of  $K_1$ . We obtain

$$\begin{pmatrix} 0 & -\frac{\rho}{1+\theta_1-\theta_\infty} \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} xK_{11} + (*)K_{21}x^{\theta_\infty-\theta_1+1} & K_{12}x^{\theta_1-\theta_\infty+1} + (*)xK_{22} \\ K_{21}x^{\theta_\infty-\theta_1+1} & xK_{22} \end{pmatrix},$$

where

$$(*) := \frac{\rho\theta_1(\theta_1 - \theta_\infty + 2) - 2r_1(\theta_1 - \theta_\infty + 1)}{2(\theta_1 - \theta_\infty)(\theta_1 - \theta_\infty + 1)}.$$

The element (2,1) must vanish. This occurs iff

$$x^{\theta_\infty-\theta_1+1} \rightarrow 0, \quad \text{for } x \rightarrow 0; \iff \text{Re}(\theta_\infty - \theta_1) > -1. \tag{88}$$

We substitute this result in the element (1, 1) and then we impose that it vanishes:

$$\frac{\rho\theta_1(\theta_1 - \theta_\infty + 2) - 2r_1(\theta_1 - \theta_\infty + 1)}{2(\theta_1 - \theta_\infty)(\theta_1 - \theta_\infty + 1)} K_{21}x^{\theta_\infty-\theta_1+1} \rightarrow 0.$$

This implies that  $\rho\theta_1(\theta_1 - \theta_\infty + 2) - 2r_1(\theta_1 - \theta_\infty + 1) = o(x^{\theta_1-\theta_\infty-1})$ .

From the element (1, 2) we have

$$-\frac{\rho}{1 + \theta_1 - \theta_\infty} \sim K_{12}x^{\theta_1-\theta_\infty+1} + o(x^{\theta_1-\theta_\infty})K_{22}. \tag{89}$$

This relation may be satisfied in two ways: the first is that  $\rho = \rho(x) = o(x^{\theta_1-\theta_\infty})$ . The second is

$$\rho = \rho(x) \sim -(1 + \theta_1 - \theta_\infty)K_{12}x^{\theta_1-\theta_\infty+1} = -rx^{\theta_1-\theta_\infty+1}. \tag{90}$$

In both cases  $\rho$  is a function of  $x$ . We are going to prove that the monodromy of  $\Psi_{\text{OUT}}$  is independent of  $\rho(x)$  (namely, of  $x$ ) if and only if

$$\rho = r_1 \frac{\theta_1 - \theta_\infty + 1}{\theta_1}. \tag{91}$$

This fact rules out the first possibility, because (89) becomes

$$\frac{\rho(x)}{\theta_\infty - \theta_1 - 1} \sim K_{12}x^{\theta_1-\theta_\infty+1} + \text{constant} \times \rho(x)x,$$

so the last term in the rhs is a higher order correction and  $\rho$  is given by (90). Before proving (91), we complete the matching procedure. Using (91), we find

$$K_0(x) \sim \begin{pmatrix} 1 & gx^{\theta_1-\theta_\infty+1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{\frac{\theta_1-\theta_\infty}{2}} & 0 \\ 0 & x^{\frac{\theta_\infty-\theta_1}{2}} \end{pmatrix}, \quad g := -\frac{r}{\theta_1 - \theta_\infty + 1}.$$

We are ready to compute  $\hat{A}_0 = K_0 \hat{A}_0 K_0^{-1}$ , where  $\hat{A}_0$  is (44), for  $\sigma = \theta_1 - \theta_\infty$ :

$$\hat{A}_0 = \begin{pmatrix} (\hat{A}_0)_{11} + g(\hat{A}_0)_{21}x & \left(\frac{(\hat{A}_0)_{12}}{x} - 2g(\hat{A}_0)_{11} - g^2(\hat{A}_0)_{21}x\right)x^{\theta_1-\theta_\infty+1} \\ \left(\frac{(\hat{A}_0)_{21}x}{x^{\theta_1-\theta_\infty+1}}\right) & -(\hat{A}_0)_{11} - g(\hat{A}_0)_{21}x \end{pmatrix}. \tag{92}$$

The first term of each matrix element certainly contains no error.

We can now substitute  $(\hat{A}_0)_{12} = \left(-\frac{r}{x} + O(1)\right)x^{\theta_1-\theta_\infty+1}$  and  $\hat{A}_1 = -r_1(x) \sim \frac{\theta_1}{\theta_1-\theta_\infty+1}rx^{\theta_1-\theta_\infty+1}$  into (3), and find

$$y(x) \sim \frac{\theta_1 - \theta_\infty + 1}{1 - \theta_\infty}, \quad x \rightarrow 0.$$

**Proof of (91).** We compute the monodromy of the solution  $\Psi_{\text{OUT}}$ . At  $\lambda = 0$  this is given by the matrix  $M_0^{\text{OUT}} = \exp\{i\pi(\theta_1 - \theta_\infty)\sigma_3\}$ , obtained by the expansion of  $\Psi_{\text{OUT}}$  at  $\lambda = 0$  as we did above, with the choice  $C_1 = e^{i\frac{\pi}{2}\theta_1} = 1/C_2$ .

Let us study the monodromy at  $\lambda = 1$ . We need to expand (86) and (87) at  $\lambda = 1$ . First of all, we use the contiguity relation

$$F(1 - \theta_1, \theta_\infty - \theta_1, \theta_\infty - \theta_1 + 1; \lambda) = \frac{\theta_1 - \theta_\infty}{\theta_\infty - 1} \frac{1}{\lambda} [(1 - \lambda)^{\theta_1} - F(1 - \theta_1, \theta_\infty - \theta_1 - 1, \theta_\infty - \theta_1; \lambda)].$$

This is used to rewrite  $\psi_1$ :

$$\psi_1 = C_1 \lambda^{\frac{\theta_1 - \theta_\infty}{2}} (\lambda - 1)^{-\frac{\theta_1}{2}} + C_2 e^{i\pi\theta_1} \lambda^{\frac{\theta_\infty - \theta_1}{2}} (\lambda - 1)^{-\frac{\theta_1}{2}} \left[ \frac{\rho - r_1}{\theta_\infty - 1} \frac{(1 - \lambda)^{\theta_1}}{\lambda} + \left( \frac{\rho}{\theta_\infty - \theta_1 - 1} + \frac{r_1 - \rho}{\theta_\infty - 1} \right) \frac{1}{\lambda} F(1 - \theta_1, \theta_\infty - \theta_1 - 1, \theta_\infty - \theta_1; \lambda) \right].$$

Then, we substitute in  $\psi_1$  the following connection formula:

$$F(1 - \theta_1, \theta_\infty - \theta_1 - 1, \theta_\infty - \theta_1; \lambda) = \frac{\Gamma(\theta_1)\Gamma(\theta_\infty - \theta_1)}{\Gamma(\theta_\infty - 1)} F(1 - \theta_1, \theta_\infty - \theta_1 - 1, 1 - \theta_1; 1 - \lambda) + \frac{\Gamma(-\theta_1)\Gamma(\theta_\infty - \theta_1)}{\Gamma(1 - \theta_1)\Gamma(\theta_\infty - \theta_1 - 1)} (1 - \lambda)^{\theta_1} F(1, \theta_\infty - 1, 1 + \theta_1; 1 - \lambda), \quad \theta_1, \theta_\infty \notin \mathbf{Z}.$$

Thus,  $\psi_1$  has the following structure, when  $\lambda \rightarrow 1$ :

$$\psi_1 = C_1 e^{-i\frac{\pi}{2}\theta_1} \lambda^{\frac{\theta_1 - \theta_\infty}{2}} (1 - \lambda)^{-\frac{\theta_1}{2}} + C_2 e^{i\frac{\pi}{2}\theta_1} \lambda^{\frac{\theta_\infty - \theta_1}{2}} \left[ (1 - \lambda)^{\frac{\theta_1}{2}} \sum_{n=0}^{\infty} a_n (1 - \lambda)^n + \left( \frac{\rho}{\theta_\infty - \theta_1 - 1} + \frac{r_1 - \rho}{\theta_\infty - 1} \right) (1 - \lambda)^{-\frac{\theta_1}{2}} \sum_{n=0}^{\infty} b_n (1 - \lambda)^n \right],$$

where  $a_n, b_n \neq 0$  are the coefficients that follow from the expansion  $1/\lambda$  and the hypergeometric functions at  $\lambda = 1$ . When  $\lambda^{\pm \frac{\theta_\infty - \theta_1}{2}}$  is also expanded at  $\lambda = 1$ , we find that  $\Psi_{\text{OUT}}$  has the following structure:

$$\Psi_{\text{OUT}} = \begin{pmatrix} (\text{series}_1)(1 - \lambda)^{-\frac{\theta_1}{2}} & (\text{series}_2)(1 - \lambda)^{\frac{\theta_1}{2}} + \left( \frac{\rho}{\theta_\infty - \theta_1 - 1} + \frac{r_1 - \rho}{\theta_\infty - 1} \right) (\text{series}_3)(1 - \lambda)^{-\frac{\theta_1}{2}} \\ 0 & (\text{series}_4)(1 - \lambda)^{\frac{\theta_1}{2}} \end{pmatrix}.$$

Here ‘series’ means a series of the form  $\sum_{n=0}^{\infty} c_n (1 - \lambda)^n$ , with  $c_0 \neq 0$ . We just give the dominant term, with the choice  $C_1 = e^{i\frac{\pi}{2}\theta_1} = 1/C_2$ :

$$\Psi_{\text{OUT}} = [I + O(1 - \lambda)] \times \begin{pmatrix} (1 - \lambda)^{-\frac{\theta_1}{2}} & \left[ \frac{\rho - r_1}{\theta_\infty - 1} + (**) \frac{\Gamma(-\theta_1)\Gamma(\theta_\infty - \theta_1)}{\Gamma(1 - \theta_1)\Gamma(\theta_\infty - \theta_1 - 1)} \right] (1 - \lambda)^{\frac{\theta_1}{2}} + (***) \frac{\Gamma(\theta_1)\Gamma(\theta_\infty - \theta_1)}{\Gamma(\theta_\infty - 1)} (1 - \lambda)^{-\frac{\theta_1}{2}} \\ 0 & (1 - \lambda)^{\frac{\theta_1}{2}} \end{pmatrix},$$

where

$$(**) = \frac{\rho}{\theta_\infty - \theta_1 - 1} + \frac{r_1 - \rho}{\theta_\infty - 1}.$$

Let  $v, w$  be non-zero arbitrary numbers. We conclude that

$$\Psi_{\text{OUT}} = \begin{pmatrix} v & w \\ 0 & w \left[ \frac{\rho - r_1}{\theta_\infty - 1} + (***) \frac{\Gamma(-\theta_1)\Gamma(\theta_\infty - \theta_1)}{\Gamma(1 - \theta_1)\Gamma(\theta_\infty - \theta_1 - 1)} \right]^{-1} \end{pmatrix} [I + O(1 - \lambda)] \times \begin{pmatrix} (1 - \lambda)^{-\frac{\theta_1}{2}} & 0 \\ 0 & (1 - \lambda)^{\frac{\theta_1}{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{v} & \frac{1}{v} (***) \frac{\Gamma(\theta_1)\Gamma(\theta_\infty - \theta_1)}{\Gamma(\theta_\infty - 1)} \\ 0 & \frac{1}{w} \left[ \frac{\rho - r_1}{\theta_\infty - 1} + (***) \frac{\Gamma(-\theta_1)\Gamma(\theta_\infty - \theta_1)}{\Gamma(1 - \theta_1)\Gamma(\theta_\infty - \theta_1 - 1)} \right] \end{pmatrix}.$$

The monodromy matrix is then

$$M_1^{\text{OUT}} = Q^{-1} e^{-i\pi\theta_1\sigma_3} Q, \quad Q := \begin{pmatrix} \frac{1}{v} & \frac{1}{v} (**) \frac{\Gamma(\theta_1)\Gamma(\theta_\infty-\theta_1)}{\Gamma(\theta_\infty-1)} \\ 0 & \frac{1}{w} \left[ \frac{\rho-r_1}{\theta_\infty-1} + (**) \frac{\Gamma(-\theta_1)\Gamma(\theta_\infty-\theta_1)}{\Gamma(1-\theta_1)\Gamma(\theta_\infty-\theta_1-1)} \right] \end{pmatrix}.$$

This is independent of  $\rho(x)$  if and only if  $(**) = 0$ . This proves (91). □

(91) simplifies considerably the structure of (87):

$$\psi_1 = C_1 \lambda^{\frac{\theta_1-\theta_\infty}{2}} (\lambda - 1)^{-\frac{\theta_1}{2}} - C_2 \frac{r_1}{\theta_1} \lambda^{\frac{\theta_\infty-\theta_1}{2}-1} (\lambda - 1)^{\frac{\theta_1}{2}}.$$

With the choice  $C_1 = \exp\{i\frac{\pi}{2}\theta_1\} = 1/C_2$ , we finally have

$$\Psi_{\text{OUT}} = \begin{pmatrix} \lambda^{\frac{\theta_1-\theta_\infty}{2}} (1-\lambda)^{-\frac{\theta_1}{2}} & -\frac{r_1(x)}{\theta_1} \lambda^{\frac{\theta_\infty-\theta_1}{2}-1} (1-\lambda)^{\frac{\theta_1}{2}} \\ 0 & \lambda^{\frac{\theta_\infty-\theta_1}{2}} (1-\lambda)^{\frac{\theta_1}{2}} \end{pmatrix}.$$

Note. The monodromy at  $\lambda = 0, 1$  is diagonal:  $M_0^{\text{OUT}} = \exp\{i\pi(\theta_1 - \theta_\infty)\sigma_3\}$ ,  $M_1^{\text{OUT}} = \exp\{-i\pi\theta_1\sigma_3\}$ . It is independent of  $r_1$ . In case (I)—namely,  $r_1$  is a non-zero constant and the system for  $\Psi_1$  is Fuchsian—if  $M_0^{\text{OUT}}$  is in diagonal form, then  $M_1^{\text{OUT}}$  is upper triangular and depends on  $r_1$ .

6.2. Higher orders terms

We may repeat the same procedure as section 5.6. We write a Taylor expansion  $y(x) = (\theta_1 - \theta_\infty + 1)/(1 - \theta_\infty) + \sum_{n \geq 1} b_n x^n$ , substitute it into (PVI) and determine recursively  $b_n$ 's. Then, we may substitute the result in the matrix elements of  $A_i$ ,  $i = 0, x, 1$ , according to the formulae of [17], and find the higher orders of the matrix elements. We just give the result:

$$\begin{aligned} (A_x)_{11} = -(A_x)_{22} &= \frac{(\theta_\infty - \theta_1)^2 + \theta_x^2 - \theta_0^2}{4(\theta_1 - \theta_\infty)} \\ &+ \frac{\theta_1 [(\theta_1 - \theta_\infty)^2 - (\theta_0 - \theta_x)^2][(\theta_1 - \theta_\infty)^2 - (\theta_0 + \theta_x)^2]}{8(\theta_1 - \theta_\infty)^2[(\theta_1 - \theta_\infty)^2 - 1]} x + O(x^2), \\ (A_x)_{12} &= -r \left\{ \frac{1}{x} - \frac{\theta_1 [(\theta_1 - \theta_\infty + 2)(\theta_1 - \theta_\infty) + \theta_0^2 - \theta_x^2]}{2(\theta_1 - \theta_\infty)(\theta_\infty - \theta_1 - 1)} + O(x) \right\} x^{\theta_1 - \theta_\infty + 1}, \\ (A_x)_{21} &= \frac{1}{r} \frac{[(\theta_1 - \theta_\infty)^2 - (\theta_0 - \theta_x)^2][(\theta_1 - \theta_\infty)^2 - (\theta_0 + \theta_x)^2]}{16(\theta_\infty - \theta_1)^2} \\ &\times \left\{ x - \frac{\theta_1 [(\theta_\infty - \theta_1)(\theta_\infty - \theta_1 + 2) + \theta_0^2 - \theta_x^2]}{2(\theta_\infty - \theta_1)^3(\theta_\infty - \theta_1 + 1)} x^2 + O(x^3) \right\} \frac{1}{x^{\theta_1 - \theta_\infty + 1}}, \\ (A_0)_{11} = -(A_0)_{22} &= \frac{(\theta_\infty - \theta_1)^2 + \theta_0^2 - \theta_x^2}{4(\theta_1 - \theta_\infty)} \\ &- \frac{\theta_1 [(\theta_1 - \theta_\infty)^2 - (\theta_0 - \theta_x)^2][(\theta_1 - \theta_\infty)^2 - (\theta_0 + \theta_x)^2]}{8(\theta_1 - \theta_\infty)^2[(\theta_1 - \theta_\infty)^2 - 1]} x + O(x^2), \\ (A_0)_{12} &= r \left\{ \frac{1}{x} - \frac{\theta_1 [(\theta_1 - \theta_\infty)^2 + \theta_0^2 - \theta_x^2]}{2(\theta_1 - \theta_\infty)(\theta_\infty - \theta_1 - 1)} + O(x) \right\} x^{\theta_1 - \theta_\infty + 1}, \\ (A_0)_{21} &= -\frac{1}{r} \frac{[(\theta_1 - \theta_\infty)^2 - (\theta_0 - \theta_x)^2][(\theta_1 - \theta_\infty)^2 - (\theta_0 + \theta_x)^2]}{16(\theta_\infty - \theta_1)^2} \\ &\times \left\{ x - \frac{\theta_1 [(\theta_1 - \theta_\infty)^2 + \theta_0^2 - \theta_x^2]}{2(\theta_\infty - \theta_1)^3(\theta_\infty - \theta_1 + 1)} x^2 + O(x^3) \right\} \frac{1}{x^{\theta_1 - \theta_\infty + 1}}, \end{aligned}$$



$$(A_1)_{11} = -(A_1)_{22} = -\frac{\theta_1}{2} - \frac{\theta_1[(\theta_1 - \theta_\infty)^2 - (\theta_0 - \theta_x)^2][(\theta_1 - \theta_\infty)^2 - (\theta_0 + \theta_x)^2]}{16[(\theta_\infty - \theta_1)^2 - 1](\theta_\infty - \theta_1)^2}x^2 + O(x^3),$$

$$(A_1)_{12} = -r \frac{\theta_1}{\theta_\infty - \theta_1 - 1} \times \left\{ 1 + \frac{(\theta_1 + 1)[(\theta_\infty - \theta_1)(\theta_\infty - \theta_1 - 2) + \theta_0^2 - \theta_x^2]}{2(\theta_\infty - \theta_1)(\theta_\infty - \theta_1 - 2)}x + O(x^2) \right\} x^{\theta_1 - \theta_\infty + 1},$$

$$(A_1)_{21} = \left\{ \frac{\theta_1[(\theta_1 - \theta_\infty)^2 - (\theta_0 - \theta_x)^2][(\theta_1 - \theta_\infty)^2 - (\theta_0 + \theta_x)^2]}{16r(\theta_\infty - \theta_1 + 1)(\theta_\infty - \theta_1)^2}x^2 \right\} \frac{1}{x^{\theta_1 - \theta_\infty + 1}}.$$

The above leading terms of  $y(x)$  are related to the above formulae through (3). The truncation of  $(A_0)_{12}$  and  $(A_1)_{12}$  above is enough to reproduce the first two terms of solution (15):

$$y(x) = \frac{\theta_1 - \theta_\infty + 1}{1 - \theta_\infty} + \frac{\theta_1[(\theta_1 - \theta_\infty)^2 + \theta_x^2 - \theta_0^2 + 2\theta_1 - 2\theta_\infty]}{2(1 - \theta_\infty)(\theta_\infty - \theta_1)(\theta_1 - \theta_\infty + 2)}x + O(x^2).$$

### 6.3. Transpose case and general result

In the above computations, the condition  $x^{\theta_\infty - \theta_1 + 1} \rightarrow 0$  was necessary to do the matching. We can repeat the matching procedure starting from the transpose matrices:

$$\hat{A}_1 = \begin{pmatrix} -\frac{\theta_1}{2} & 0 \\ -r_1 & \frac{\theta_1}{2} \end{pmatrix}, \quad \hat{A}_0 + \hat{A}_x = \begin{pmatrix} \frac{\theta_1 - \theta_\infty}{2} & 0 \\ r_1 & -\frac{\theta_1 - \theta_\infty}{2} \end{pmatrix}, \quad xA_x = \begin{pmatrix} 0 & 0 \\ \rho(x) & 0 \end{pmatrix}.$$

The procedure is exactly the same, with the necessary condition:

$$x^{\theta_1 - \theta_\infty + 1} \rightarrow 0, \quad \text{when } x \rightarrow 0.$$

As a result, we again obtain exactly the matrices  $A_0$ ,  $A_x$ ,  $A_1$  above. The reader can convince himself without doing any computation, simply looking at the structure of the first terms of above matrices. For example, let us have a look at  $A_x$ . Denote the constant terms with letters  $c_1, c_2, \dots$ , etc. We have

$$A_x = \begin{pmatrix} c_1 + \dots & \left\{ \frac{c_2}{x} + \dots \right\} x^{\theta_1 - \theta_\infty + 1} \\ \left\{ c_3x + \dots \right\} \frac{1}{x^{\theta_1 - \theta_\infty + 1}} & -c_1 + \dots \end{pmatrix} = \begin{pmatrix} c_1 + \dots & \left\{ c_2x + \dots \right\} \frac{1}{x^{\theta_\infty - \theta_1 + 1}} \\ \left\{ \frac{c_3}{x} + \dots \right\} x^{\theta_\infty - \theta_1 + 1} & -c_1 + \dots \end{pmatrix}.$$

The role of  $\theta_\infty - \theta_1 + 1$  and  $\theta_1 - \theta_\infty + 1$  is just exchanged. By continuity, the matrices  $A_0$ ,  $A_x$ ,  $A_1$  computed above hold for any value of  $\theta_1 - \theta_\infty \notin \mathbf{Z}$ ,  $\theta_\infty \neq 1$ .

The matching procedure can be repeated in the same way in the case  $\sigma = -(\theta_\infty - \theta_1)$  and in the case  $\sigma = \pm(\theta_\infty + \theta_1)$ , which yields (18). For these last cases, the results are obtained just by the substitution  $\theta_1 \mapsto -\theta_1$ .

### 6.4. Monodromy data

We compute the monodromy data for the case  $\sigma = \theta_1 - \theta_\infty$ , all the other cases being analogous. In this case, the matching has been realized by

$$\Psi_{\text{OUT}}(\lambda, x) := \begin{pmatrix} \lambda^{\frac{\theta_1 - \theta_\infty}{2}} (1 - \lambda)^{-\frac{\theta_1}{2}} & -\frac{r_1(x)}{\theta_1} \lambda^{\frac{\theta_\infty - \theta_1}{2} - 1} (1 - \lambda)^{\frac{\theta_1}{2}} \\ 0 & \lambda^{\frac{\theta_\infty - \theta_1}{2}} (1 - \lambda)^{\frac{\theta_1}{2}} \end{pmatrix}.$$

$$\Psi_{\text{IN}}(\lambda, x) = K_0(x) \Psi_0 \left( \frac{\lambda}{x} \right), \quad \Psi_0(\mu) = \left[ I + O \left( \frac{1}{\mu} \right) \right] \mu^{\frac{\theta_1 - \theta_\infty}{2} \sigma_3}, \quad \mu \rightarrow \infty.$$

Let  $\Psi(\lambda)$  denote the solution of system (1) of (PVI), such that  $\Psi(\lambda) = [I + O(\lambda^{-1})]\lambda^{-\frac{\theta_\infty}{2}\sigma_3}$ ,  $\lambda \rightarrow \infty, \theta_\infty \notin \mathbf{Z}$ .

Matching  $\Psi \leftrightarrow \Psi_{\text{OUT}}$

With the choice  $1 - \lambda = e^{-i\pi}(\lambda - 1)$  ( $1 - \lambda > 0$  for  $0 < \lambda < 1$ ) we have

$$\Psi_{\text{OUT}} = \left[ I + O\left(\frac{1}{\lambda}\right) \right] \lambda^{-\frac{\theta_\infty}{2}\sigma_3} e^{i\frac{\pi}{2}\theta_1\sigma_3}, \quad \lambda \rightarrow \infty$$

Therefore, the correct choice for  $\Psi_{\text{OUT}}$ , which matches with  $\Psi$  is

$$\Psi_{\text{OUT}}^{\text{Match}} := \Psi_{\text{OUT}} C_{\text{OUT}}, \quad C_{\text{OUT}} := e^{-i\frac{\pi}{2}\theta_1\sigma_3}.$$

As a consequence, we obtain

(i) the monodromy of  $\Psi$  at  $\lambda = 1, \infty$ , which is equal to the monodromy of  $\Psi_{\text{OUT}}^{\text{Match}}$ :

$$M_1 = e^{-i\pi\theta_1\sigma_3}, \quad M_\infty = e^{-i\pi\theta_\infty\sigma_3};$$

(ii) the correct choice for  $\Psi_{\text{IN}}^{\text{Match}} := K_0(x)\Psi_0 C_{\text{OUT}}$ .

Matching  $\Psi \leftrightarrow \Psi_{\text{IN}}^{\text{Match}}$

This is realized by construction. As a consequence, we can compute the monodromy of  $\Psi$  at  $\lambda = 0, x$ . In order to do this, we need the local behaviour of  $\Psi_{\text{IN}}^{\text{Match}}(\lambda, x)$  at  $\lambda = 0, x$ . We start with  $\Psi_0$ , recalling that

$$\Psi_0(\mu) = \mu^{\frac{\theta_0}{2}}(\mu - 1)^{\frac{\theta_x}{2}}\Phi_0(\mu), \quad \Phi_0(\mu) = \begin{pmatrix} \varphi_1(\mu) & \varphi_2(\mu) \\ \xi_1(\mu) & \xi_2(\mu) \end{pmatrix}.$$

Here,  $\varphi_1, \varphi_2$  are two independent solutions of a Gauss hyper-geometric equation (see (A.11) in appendix A):

$$\mu(1 - \mu) \frac{d^2\varphi}{d\mu^2} + (1 + c - (a + [b + 1] + 1)\mu) \frac{d\varphi}{d\mu} - a(b + 1)\varphi = 0,$$

where  $a := \frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2}, b + 1 := \frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1, c + 1 := \theta_0 + 1$ . The functions  $\xi_1$  and  $\xi_2$  are obtained from  $\varphi_1$  and  $\varphi_2$  by

$$\xi = \frac{1}{r} \left[ \mu(1 - \mu) \frac{d\varphi}{d\mu} - a \left( \mu + \frac{b - c}{a - b} \right) \varphi \right]. \tag{93}$$

In order to have generic solutions (i.e., non-logarithmic solutions) of the hyper-geometric equation, we must require

$$\theta_1 - \theta_\infty, \theta_0, \theta_x \notin \mathbf{Z}.$$

Then, we have the following sets of independent solutions at  $\mu = 0, 1, \infty$ , respectively (we denote by  $F$  the Gauss hyper-geometric function):

$$\begin{cases} \varphi_1^{(0)} = F\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2}, \frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1, 1 + \theta_0; \mu\right), \\ \varphi_2^{(0)} = \mu^{-\theta_0} F\left(\frac{\theta_x}{2} - \frac{\theta_0}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2}, \frac{\theta_x}{2} - \frac{\theta_0}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1, 1 - \theta_0; \mu\right). \\ \varphi_1^{(1)} = F\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2}, \frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1, 1 + \theta_x; 1 - \mu\right), \\ \varphi_2^{(1)} = (1 - \mu)^{-\theta_x} F\left(\frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2}, \frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1, 1 - \theta_x; 1 - \mu\right). \\ \varphi_1^{(\infty)} = \mu^{-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{\theta_\infty}{2} + \frac{\theta_1}{2}} F\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2}, \frac{\theta_x}{2} - \frac{\theta_0}{2} + \frac{\theta_\infty}{2} - \frac{\theta_1}{2}, \theta_\infty - \theta_1; \frac{1}{\mu}\right), \\ \varphi_2^{(\infty)} = \mu^{-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{\theta_1}{2} + \frac{\theta_\infty}{2} - 1} F\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1, \frac{\theta_x}{2} - \frac{\theta_0}{2} + \frac{\theta_1}{2} - \frac{\theta_\infty}{2} + 1, 2 + \theta_1 - \theta_\infty; \frac{1}{\mu}\right). \end{cases}$$

The connection formulae can be found in any book on special functions:

$$\begin{aligned} [\varphi_1^{(0)}, \varphi_2^{(0)}] &= [\varphi_1^{(1)}, \varphi_2^{(1)}] C_{01}, & -\pi < \arg(1 - \mu) < \pi. \\ [\varphi_1^{(0)}, \varphi_2^{(0)}] &= [\varphi_1^{(\infty)}, \varphi_2^{(\infty)}] C_{0\infty}, & 0 < \arg \mu < 2\pi. \end{aligned}$$

Here, the connection matrices  $C_{01}$ ,  $C_{0\infty}$  are (31) and (30), respectively.

From the Taylor expansion of the hyper-geometric functions in  $\varphi_i^{(\infty)}$  and (93), we compute:

$$\begin{aligned}\mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\varphi_1^{(\infty)} &= \left[1 + O\left(\frac{1}{\mu}\right)\right] \mu^{\frac{\theta_1-\theta_{\infty}}{2}}, \\ \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\varphi_2^{(\infty)} &= \frac{1}{\mu} \left[1 + O\left(\frac{1}{\mu}\right)\right] \mu^{\frac{\theta_{\infty}-\theta_1}{2}}, \\ \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\xi_1^{(\infty)} &= \left[1 + O\left(\frac{1}{\mu}\right)\right] \mu^{\frac{\theta_1-\theta_{\infty}}{2}}, \\ \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\xi_2^{(\infty)} &= \frac{\theta_1-\theta_{\infty}+1}{r_0} \left[1 + O\left(\frac{1}{\mu}\right)\right] \mu^{\frac{\theta_{\infty}-\theta_1}{2}}.\end{aligned}$$

It follows that the matrix  $\Psi_0(\mu) = \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\Phi_0(\mu)$  with the prescribed behaviour  $\left[I + O\left(\frac{1}{\mu}\right)\right]\mu^{\frac{\theta_1-\theta_{\infty}}{2}\sigma_3}$  at  $\mu = \infty$  is

$$\Psi_0(\mu) = \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(\infty)} & \varphi_2^{(\infty)} \\ \xi_1^{(\infty)} & \xi_2^{(\infty)} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{r}{\theta_1-\theta_{\infty}+1} \end{pmatrix}.$$

Let

$$C := \begin{pmatrix} 1 & 0 \\ 0 & \frac{r}{\theta_1-\theta_{\infty}+1} \end{pmatrix} C_{\text{OUT}} \equiv \begin{pmatrix} 1 & 0 \\ 0 & \frac{r}{\theta_1-\theta_{\infty}+1} \end{pmatrix} e^{-i\frac{\pi}{2}\theta_1\sigma_3}.$$

We conclude that

$$\begin{aligned}\Psi_{\text{IN}}^{\text{Match}}(\lambda, x) &= K_0(x) \begin{pmatrix} \frac{\lambda}{x} \\ \frac{\lambda}{x} \end{pmatrix}^{\frac{\theta_0}{2}} \begin{pmatrix} \frac{\lambda}{x} - 1 \\ \frac{\lambda}{x} - 1 \end{pmatrix}^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(\infty)}\left(\frac{\lambda}{x}\right) & \varphi_2^{(\infty)}\left(\frac{\lambda}{x}\right) \\ \xi_1^{(\infty)}\left(\frac{\lambda}{x}\right) & \xi_2^{(\infty)}\left(\frac{\lambda}{x}\right) \end{pmatrix} C, \\ &= K_0(x) \begin{pmatrix} \frac{\lambda}{x} \\ \frac{\lambda}{x} \end{pmatrix}^{\frac{\theta_0}{2}} \begin{pmatrix} \frac{\lambda}{x} - 1 \\ \frac{\lambda}{x} - 1 \end{pmatrix}^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(0)}\left(\frac{\lambda}{x}\right) & \varphi_2^{(0)}\left(\frac{\lambda}{x}\right) \\ \xi_1^{(0)}\left(\frac{\lambda}{x}\right) & \xi_2^{(0)}\left(\frac{\lambda}{x}\right) \end{pmatrix} C_{0\infty}^{-1} C, \\ &= K_0(x) \begin{pmatrix} \frac{\lambda}{x} \\ \frac{\lambda}{x} \end{pmatrix}^{\frac{\theta_0}{2}} \begin{pmatrix} \frac{\lambda}{x} - 1 \\ \frac{\lambda}{x} - 1 \end{pmatrix}^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(1)}\left(\frac{\lambda}{x}\right) & \varphi_2^{(1)}\left(\frac{\lambda}{x}\right) \\ \xi_1^{(1)}\left(\frac{\lambda}{x}\right) & \xi_2^{(1)}\left(\frac{\lambda}{x}\right) \end{pmatrix} C_{01} C_{0\infty}^{-1} C.\end{aligned}$$

The behaviours of the above matrix at  $\lambda = x, 0$  are easily computed from

$$\begin{aligned}\mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\varphi_1^{(1)} &= \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}} F(\dots; 1-\mu) = x^{-\frac{\theta_0}{2}-\frac{\theta_x}{2}} F\left(\dots; \frac{x-\lambda}{x}\right) \lambda^{\frac{\theta_0}{2}} (\lambda-x)^{\frac{\theta_x}{2}}, \\ \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\varphi_2^{(1)} &= \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}} (1-\mu)^{-\frac{\theta_x}{2}} = (-1)^{\theta_x} x^{-\frac{\theta_0}{2}+\frac{\theta_x}{2}} F\left(\dots; \frac{x-\lambda}{x}\right) \lambda^{\frac{\theta_0}{2}} (\lambda-x)^{-\frac{\theta_x}{2}},\end{aligned}$$

and

$$\begin{aligned}\mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\varphi_1^{(0)} &= \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}} F(\dots; \mu) = x^{-\frac{\theta_0}{2}-\frac{\theta_x}{2}} F\left(\dots; \frac{\lambda}{x}\right) (\lambda-x)^{\frac{\theta_x}{2}} \lambda^{\frac{\theta_0}{2}}, \\ \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}}\varphi_2^{(0)} &= \mu^{-\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}} F(\dots; \mu) = x^{\frac{\theta_0}{2}-\frac{\theta_x}{2}} F\left(\dots; \frac{\lambda}{x}\right) (\lambda-x)^{\frac{\theta_x}{2}} \lambda^{-\frac{\theta_0}{2}}.\end{aligned}$$

From these we compute

$$\begin{aligned}K_0(x) \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(0)} & \varphi_2^{(0)} \\ \xi_1^{(0)} & \xi_2^{(0)} \end{pmatrix} &= \psi_0^{\text{IN}}(x) (I + O(\lambda)) \lambda^{\frac{\theta_0}{2}\sigma_3}, & \lambda \rightarrow 0 \\ K_0(x) \mu^{\frac{\theta_0}{2}}(\mu-1)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} \\ \xi_1^{(1)} & \xi_2^{(1)} \end{pmatrix} &= \psi_x^{\text{IN}}(x) (I + O(\lambda-x)) (\lambda-x)^{\frac{\theta_x}{2}\sigma_3}, & \lambda \rightarrow x.\end{aligned}$$

Here, we do not need to explicitly give the invertible matrices  $\psi_0^{\text{IN}}(x)$  and  $\psi_x^{\text{IN}}(x)$ . From the above procedure, we find

$$M_0 = C^{-1}(C_{0\infty} \exp\{i\pi\theta_0\sigma_3\}C_{0\infty}^{-1})C, \quad M_x = C^{-1}[C_{0\infty}(C_{01}^{-1} \exp\{i\pi\theta_x\sigma_3\}C_{01})C_{0\infty}^{-1}]C.$$

We finally observe that  $M_1$  and  $M_\infty$  are diagonal, so they are invariant for the conjugation  $M \mapsto CMC^{-1}$ . With this in mind, we get the result of theorem 3, point (a).

We stress that  $\hat{A}_0 + \hat{A}_x$  and  $\hat{A}_1$  are upper (or lower) triangular matrices, and the group generated by  $M_x M_0$  and  $M_1$  is reducible.

**7. Case  $\theta_\infty = 1, \theta_1 = 0, \sigma = \pm 1$ : solution (17)**

In section 6, we imposed that  $\theta_\infty \neq 1$  and  $\theta_\infty - \theta_1 \notin \mathbf{Z}$ . Here, we consider the case  $\theta_\infty = 1, \theta_\infty - \theta_1 = 1$ . We have

$$\hat{A}_0 + \hat{A}_x = \begin{pmatrix} -\frac{1}{2} & r_1 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad \hat{A}_1 = \begin{pmatrix} 0 & -r_1 \\ 0 & 0 \end{pmatrix}.$$

Also the transpose matrices are possible. For  $|\lambda| \leq |x|^{\delta_{\text{IN}}}$ , we use the reduction (43) (namely, (11)). The matrices  $\hat{A}_0, \hat{A}_x$  are given by (44) and (45) with the substitution  $\sigma = 1$  or  $-1$ . For definiteness, let us take  $\sigma = 1$  in the following.

For  $|\lambda| \geq |x|^{\delta_{\text{OUT}}}$ , we approximate (1) with  $\frac{d\Psi_{\text{OUT}}}{d\lambda} = \left[ \frac{x A_x}{\lambda^2} + \frac{A_0 + A_x}{\lambda} + \frac{A_1}{\lambda - 1} \right] \Psi_{\text{OUT}}$ . For definiteness, let us consider the case when  $\hat{A}_1$  and  $\hat{A}_0 + \hat{A}_x$  are upper triangular. Again, we make the hypothesis that the leading terms in  $x A_x$  define an upper triangular matrix:

$$x A_x =: \begin{pmatrix} 0 & \rho \\ 0 & 0 \end{pmatrix} + \text{higher orders}.$$

Therefore, we will study

$$\frac{d\Psi_{\text{OUT}}}{d\lambda} = \left[ \frac{1}{\lambda^2} \begin{pmatrix} 0 & \rho \\ 0 & 0 \end{pmatrix} + \frac{1}{\lambda} \begin{pmatrix} -\frac{1}{2} & r_1 \\ 0 & \frac{1}{2} \end{pmatrix} + \frac{1}{\lambda - 1} \begin{pmatrix} 0 & -r_1 \\ 0 & 0 \end{pmatrix} \right] \Psi_{\text{OUT}}.$$

This is a reducible system. To solve it, we write  $\Psi_{\text{OUT}}$  in vector notation  $\Psi_{\text{OUT}} = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ . The system becomes

$$\frac{d\psi_1}{d\lambda} = -\frac{1}{2\lambda}\psi_1 + \left( \frac{\rho}{\lambda^2} + \frac{r_1}{\lambda} - \frac{r_1}{\lambda - 1} \right) \psi_2, \quad \frac{d\psi_2}{d\lambda} = \frac{1}{2\lambda}\psi_2.$$

The solution obtained by variation of parameters is

$$\psi_1 = C_1 \lambda^{-\frac{1}{2}} + C_2 \lambda^{-\frac{1}{2}} (\rho \ln \lambda - r_1 \ln(\lambda - 1)), \quad \psi_2 = C_2 \lambda^{\frac{1}{2}}, \quad C_1, C_2 \in \mathbf{C}.$$

We can choose the following fundamental matrix:

$$\Psi_{\text{OUT}} = \begin{pmatrix} \lambda^{-\frac{1}{2}} & \lambda^{-\frac{1}{2}}(\rho \ln \lambda - r_1 \ln(\lambda - 1)) \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \rho \ln \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -r_1 \ln(\lambda - 1) \\ 0 & 1 \end{pmatrix}.$$

Its monodromy relative to the loops  $\lambda \mapsto \lambda e^{2\pi i}$  and  $(\lambda - 1) \mapsto (\lambda - 1) e^{2\pi i}$  is, respectively,

$$\Psi_{\text{OUT}} \mapsto \Psi_{\text{OUT}} \begin{pmatrix} -1 & -2\pi i \rho \\ 0 & -1 \end{pmatrix}, \quad \Psi_{\text{OUT}} \mapsto \Psi_{\text{OUT}} \begin{pmatrix} 1 & -2\pi i r_1 \\ 0 & 1 \end{pmatrix}.$$

Therefore,  $\rho$  and  $r_1$  must be independent of (the monodromy-preserving deformation)  $x$ . We observe that any fundamental matrix solution of the form

$$\Psi_{\text{OUT}} \begin{pmatrix} 1 & f(x) \\ 0 & 1 \end{pmatrix}$$

has the same monodromy of  $\Psi_{\text{OUT}}$ , for any arbitrary function of  $x$ . This fact will be used soon, with the choice  $f(x) = -\rho \ln x$ .

With the choice of the branch of  $\ln(\lambda-1) = \ln(1-\lambda)+i\pi$  (i.e.,  $(\lambda-1) = e^{i\pi}(1-\lambda)$ ,  $1-\lambda > 0$  for  $0 < \lambda < 1$ ), it is convenient to redefine  $\Psi_{\text{OUT}}$  by

$$\begin{aligned} \Psi_{\text{OUT}} &:= \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \rho \ln \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -r_1 \ln(\lambda-1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i\pi r_1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\frac{r_1}{\lambda} \ln(1-\lambda) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \rho \ln \lambda \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Therefore,

$$\Psi_{\text{OUT}} = \left[ \begin{pmatrix} 1 & r_1 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & i\pi r_1 \\ 0 & 0 \end{pmatrix} \sum_{n=1}^{\infty} \frac{\lambda^n}{n+1} \right] \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \rho \ln \lambda \\ 0 & 1 \end{pmatrix}, \quad \lambda \rightarrow 0$$

### 7.1. Matching

The solution  $\Psi_0\left(\frac{\lambda}{x}\right)$  has been introduced in section 4.7:

$$\Psi_0\left(\frac{\lambda}{x}\right) = \left[ I + O\left(\frac{x}{\lambda}\right) \right] \begin{pmatrix} \left(\frac{\lambda}{x}\right)^{\frac{1}{2}} & 0 \\ 0 & \left(\frac{\lambda}{x}\right)^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R \ln\left(\frac{\lambda}{x}\right) & 1 \end{pmatrix}, \quad R := (\hat{A}_x)_{21}.$$

Some adjustments are necessary. Let us consider the permutation matrix  $P := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and redefine

$$\Psi_0 \mapsto \Psi_0 P \begin{pmatrix} 1 & 0 \\ 0 & R^{-1} \end{pmatrix}, \quad \Psi_{\text{OUT}} \mapsto \Psi_{\text{OUT}} \begin{pmatrix} 1 & -\rho \ln x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \rho^{-1} \end{pmatrix}.$$

As we have already observed, this redefinition does not affect the monodromy of  $\Psi_{\text{OUT}}$ , which is independent of  $x$ . The matching relation  $\Psi_{\text{OUT}}(\lambda) \sim K_0(x)\Psi_0(\lambda/x)$  becomes

$$\begin{aligned} &\begin{pmatrix} 1 & r_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix} \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \ln \frac{\lambda}{x} \\ 0 & 1 \end{pmatrix} \\ &\sim K_0(x) P \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{R} \end{pmatrix} \begin{pmatrix} x^{\frac{1}{2}} & 0 \\ 0 & x^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \ln \frac{\lambda}{x} \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

The matching is thus realized, with the choice

$$K_0(x) \sim \begin{pmatrix} 1 & \frac{r_1}{\rho} \\ 0 & \frac{1}{\rho} \end{pmatrix} \begin{pmatrix} x^{-\frac{1}{2}} & 0 \\ 0 & R x^{\frac{1}{2}} \end{pmatrix} P \equiv \begin{pmatrix} \frac{r_1}{\rho} & 1 \\ \frac{1}{\rho} & 0 \end{pmatrix} \begin{pmatrix} R x^{\frac{1}{2}} & 0 \\ 0 & x^{-\frac{1}{2}} \end{pmatrix}.$$

It follows that

$$\begin{aligned} \hat{A}_0 &= \begin{pmatrix} \frac{\theta_x^2 - \theta_0^2 - 1}{4} + \frac{r_1}{\rho} \frac{[1 - (\theta_0 - \theta_x)^2][1 - (\theta_0 + \theta_x)^2]}{16} x & -\frac{\rho}{x} + r_1 \frac{\theta_0^2 - \theta_x^2 + 1}{2} - \frac{r_1^2}{\rho} \frac{[1 - (\theta_0 - \theta_x)^2][1 - (\theta_0 + \theta_x)^2]}{16} x \\ \frac{1}{\rho} \frac{[1 - (\theta_0 - \theta_x)^2][1 - (\theta_0 + \theta_x)^2]}{16} x & -\frac{\theta_x^2 - \theta_0^2 - 1}{4} - \frac{r_1}{\rho} \frac{[1 - (\theta_0 - \theta_x)^2][1 - (\theta_0 + \theta_x)^2]}{16} x \end{pmatrix}, \\ \hat{A}_x &= \begin{pmatrix} \frac{\theta_0^2 - \theta_x^2 - 1}{4} - \frac{r_1}{\rho} \frac{[1 - (\theta_0 - \theta_x)^2][1 - (\theta_0 + \theta_x)^2]}{16} x & \frac{\rho}{x} + r_1 \frac{\theta_x^2 - \theta_0^2 + 1}{2} + \frac{r_1^2}{\rho} \frac{[1 - (\theta_0 - \theta_x)^2][1 - (\theta_0 + \theta_x)^2]}{16} x \\ -\frac{1}{\rho} \frac{[1 - (\theta_0 - \theta_x)^2][1 - (\theta_0 + \theta_x)^2]}{16} x & -\frac{\theta_0^2 - \theta_x^2 - 1}{4} + \frac{r_1}{\rho} \frac{[1 - (\theta_0 - \theta_x)^2][1 - (\theta_0 + \theta_x)^2]}{16} x \end{pmatrix}. \end{aligned}$$

The leading term of each matrix element certainly contains no error. Note that  $r$  has simplified. Actually, the constant  $\rho$  plays the role of  $r$ .

We will not repeat again the discussion for the higher order terms. The matrix elements are Taylor expansions, corresponding to a Taylor expanded  $y(x)$ , the convergence of which

is proved by a Briot–Bouquet argument. The first two leading terms of  $A_0$  and  $A_x$  above are actually correct. In particular, we need

$$(A_0)_{12} = -\frac{\rho}{x} + r_1 \frac{\theta_0^2 - \theta_x^2 + 1}{2} + O(x).$$

The leading terms of  $A_1$  are

$$A_1 = \begin{pmatrix} \frac{r_1}{\rho} \frac{[1-(\theta_0-\theta_x)^2][1-(\theta_0+\theta_x)^2]}{32} x^2 + O(x^3) & -r_1 + r_1 \frac{\theta_0^2 - \theta_x^2 - 1}{2} x + O(x^2) \\ \frac{r_1}{\rho^2} \frac{[1-(\theta_0-\theta_x)^2]^2 [1-(\theta_0+\theta_x)^2]^2}{1024} x^4 + O(x^5) & -\frac{r_1}{\rho} \frac{[1-(\theta_0-\theta_x)^2][1-(\theta_0+\theta_x)^2]}{32} x^2 + O(x^3) \end{pmatrix}.$$

The above truncations of  $(A_1)_{12}$  and  $(A_0)_{12}$  correspond to the first two terms of the Taylor expansion of solution (17), through (3):

$$y(x) = a + \frac{1-a}{2} (1 + \theta_0^2 - \theta_x^2)x + O(x^2), \quad \text{where } a := \left(1 - \frac{r_1}{\rho}\right)^{-1}. \tag{94}$$

Observe that  $a$  depends on the monodromy datum  $\frac{r_1}{\rho}$ .

### 7.2. Monodromy

The matching  $\Psi_{\text{IN}} \leftrightarrow \Psi_{\text{OUT}}$  has been realized by

$$\Psi_{\text{OUT}} = \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \rho \ln \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -r_1 \ln(\lambda - 1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\ln x + i\pi \frac{r_1}{\rho} \\ 0 & \frac{1}{\rho} \end{pmatrix},$$

and

$$\Psi_{\text{IN}}(\mu) = K_0(x) \Psi_0(\mu) \begin{pmatrix} 0 & R^{-1} \\ 1 & 0 \end{pmatrix}, \quad R = \frac{[(\theta_0 + \theta_x)^2 - 1][(\theta_0 - \theta_x)^2 - 1]}{16r}.$$

$$\Psi_0(\mu) = \left[ I + O\left(\frac{1}{\mu}\right) \right] \begin{pmatrix} \mu^{\frac{1}{2}} & 0 \\ 0 & \mu^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ R \ln \mu & 1 \end{pmatrix}, \quad \mu \rightarrow \infty.$$

Namely,

$$\begin{aligned} \Psi_{\text{IN}} &\sim K_0(x) P \begin{pmatrix} x^{\frac{1}{2}} & 0 \\ 0 & R^{-1} x^{-\frac{1}{2}} \end{pmatrix} \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \ln \frac{\lambda}{x} \\ 0 & 1 \end{pmatrix}, \\ &= \begin{pmatrix} 1 & \frac{r_1}{\rho} \\ 0 & \frac{1}{\rho} \end{pmatrix} \begin{pmatrix} \lambda^{-\frac{1}{2}} & 0 \\ 0 & \lambda^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} 1 & \ln \frac{\lambda}{x} \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

for  $\mu = \lambda/x \rightarrow \infty$ ,  $|\lambda| < |x|^{\delta_{\text{IN}}}$ ,  $x \rightarrow 0$ .

#### Matching $\Psi \leftrightarrow \Psi_{\text{OUT}}$

The correct choice of  $\Psi_{\text{OUT}}$  must match with

$$\Psi = \left[ I + O\left(\frac{1}{\lambda}\right) \right] \lambda^{-\frac{\theta_\infty}{2} \sigma_3} \lambda^L, \quad \lambda \rightarrow \infty, \quad \theta_\infty = 1,$$

where  $L = \begin{pmatrix} 0 & \rho - r_1 \\ 0 & 0 \end{pmatrix}$ . This form of  $L$  follows from the standard theory of Fuchsian systems and from the expansion of system (1) at  $v := \frac{1}{\lambda} \rightarrow 0$ :

$$\frac{d\Psi}{dv} = \left[ \frac{\sigma_3}{2} \frac{1}{v} - (A_1 + x A_x) + O(v) \right] \Psi, \quad v \rightarrow \infty.$$

Thus,  $L_{12} = -(A_1 + x A_x)_{1,2}|_{x=0} \equiv r_1 - \rho$  (this is computed from the expansions of  $A_1$  and  $A_x$  at  $x = 0$  obtained before). We expand  $\Psi_{\text{OUT}}$  at  $\lambda = \infty$ . We easily get

$$\Psi_{\text{OUT}} = [I + O(\lambda^{-1})] \lambda^{-\frac{1}{2} \sigma_3} \begin{pmatrix} 1 & (\rho - r_1) \ln \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & i\pi \frac{r_1}{\rho} - \ln x \\ 0 & \frac{1}{\rho} \end{pmatrix}.$$

Therefore, the correct choice is

$$\Psi_{\text{OUT}}^{\text{Match}} := \Psi_{\text{OUT}} \begin{pmatrix} 1 & i\pi \frac{r_1}{\rho} - \ln x \\ 0 & \frac{1}{\rho} \end{pmatrix}^{-1} = \lambda^{-\frac{1}{2}\sigma_3} \begin{pmatrix} 1 & \rho \ln \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -r_1 \ln(\lambda - 1) \\ 0 & 1 \end{pmatrix}.$$

As a consequence, we obtain the following:

- (i) The monodromy of  $\Psi$  at  $\lambda = 1, \infty$ . This coincides with that of  $\Psi_{\text{OUT}}^{\text{Match}}$ , which is easily computed from the local behaviour:

$$\Psi_{\text{OUT}}^{\text{Match}}(\lambda) = [I + O(\lambda - 1)] \begin{pmatrix} 1 & -r_1 \ln(\lambda - 1) \\ 0 & 1 \end{pmatrix}, \quad \lambda \rightarrow 1.$$

Thus, for  $(\lambda - 1) \mapsto (\lambda - 1) \exp\{2\pi i\}$  and  $\lambda \mapsto \lambda \exp\{2\pi i\}$  the monodromy is

$$M_1 = \begin{pmatrix} 1 & -2\pi i r_1 \\ 0 & 1 \end{pmatrix}, \quad M_\infty = \begin{pmatrix} -1 & 2\pi i(r_1 - \rho) \\ 0 & -1 \end{pmatrix}.$$

- (ii) The correct choice of  $\Psi_{\text{IN}}$ , which matches with  $\Psi$ . This is

$$\begin{aligned} \Psi_{\text{IN}}^{\text{Match}}(\lambda, x) &= \Psi_{\text{IN}}(\lambda, x) \begin{pmatrix} 1 & i\pi \frac{r_1}{\rho} - \ln x \\ 0 & \frac{1}{\rho} \end{pmatrix}^{-1} = \Psi_{\text{IN}}(\lambda, x) \begin{pmatrix} 1 & \rho \ln x - i\pi r_1 \\ 0 & \rho \end{pmatrix} \\ &= K_0(x) \Psi_0(\mu) C_{\text{IN}}, \quad C_{\text{IN}} = \begin{pmatrix} 0 & \rho/R \\ 1 & \rho \ln x - i\pi r_1 \end{pmatrix}. \end{aligned}$$

**Important remark.**  $C_{\text{IN}}$  depends on  $x$ . But the monodromy must be independent of  $x$ . For this reason, we will have to consider the substitution

$$\Psi_{\text{IN}}^{\text{Match}} \mapsto \Psi_{\text{IN}}^{\text{Match}}(C_{\text{IN}})^{-1} = K_0(x) \Psi_0(\mu).$$

This makes the monodromy at  $\lambda = 0, x$  independent of  $x$ . The corresponding transformation,

$$\Psi_{\text{OUT}}^{\text{Match}} \mapsto \Psi_{\text{OUT}}^{\text{Match}} C_{\text{IN}}^{-1},$$

changes  $M_1, M_\infty$ , but does not introduce a dependence on  $x$ . Namely,

$$\begin{aligned} M_1 &\mapsto C_{\text{IN}} M_1 C_{\text{IN}}^{-1} = \begin{pmatrix} 1 & 0 \\ -2\pi i \frac{r_1}{\rho} R & 1 \end{pmatrix}, \\ M_\infty &\mapsto C_{\text{IN}} M_\infty C_{\text{IN}}^{-1} = \begin{pmatrix} -1 & 0 \\ 2\pi i \left(\frac{r_1}{\rho} - 1\right) R & -1 \end{pmatrix}. \end{aligned}$$

*Matching*  $\Psi \leftrightarrow \Psi_{\text{IN}}$

$\Psi$  and  $\Psi_{\text{IN}}^{\text{Match}}(\lambda, x)$  are matching by construction. The monodromy of  $\Psi$  at  $\lambda = 0, x$  coincides with that of  $\Psi_{\text{IN}}^{\text{Match}}(\lambda, x)$ . In order to compute it, we write

$$\Psi_0(\mu) = \mu^{\frac{\theta_0}{2}} (\mu - 1)^{\frac{\theta_x}{2}} \Phi_0(\mu), \quad \Phi_0(\mu) = \begin{pmatrix} \varphi_1 & \varphi_2 \\ \xi_1 & \xi_2 \end{pmatrix}.$$

$\Phi_0$  satisfies (46) with  $\sigma = 1$ . It is expressed in terms of two independent solutions  $\varphi_1, \varphi_2$  of the Gauss hyper-geometric equation (see appendix A):

$$\mu(1 - \mu) \frac{d^2 \varphi}{d\mu^2} + (1 + c - (a + [b + 1] + 1)\mu) \frac{d\varphi}{d\mu} - a(b + 1)\varphi = 0,$$

where

$$a = \frac{\theta_0}{2} + \frac{\theta_x}{2} - \frac{1}{2}, \quad b + 1 = \frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}, \quad c + 1 = \theta_0 + 1.$$

From  $\varphi_1$  and  $\varphi_2$  we compute

$$\xi_i = \frac{1}{r} \left[ \mu(1 - \mu) \frac{d\varphi_i}{d\mu} - a \left( \mu + \frac{b - c}{a - b} \right) \varphi_i \right], \quad i = 1, 2.$$

We need a complete set of solutions at  $\mu = 0, 1, \infty$ . Since  $a - b$  is an integer, we have a logarithmic case. We briefly explain some preliminary facts. Let us consider a Gauss Hyper-geometric equation in standard form

$$\mu(1 - \mu) \frac{d^2\varphi}{d\mu^2} + [\gamma - (\alpha + \beta + 1)\mu] \frac{d\varphi}{d\mu} - \alpha\beta\varphi = 0$$

( $\alpha, \beta, \gamma$  here are not the coefficients of (PVI)! We are just using the same symbols only here). Logarithmic solutions at  $\mu = 0$  may occur only if  $\gamma \in \mathbf{Z}$ , at  $\mu = 1$  if  $\alpha + \beta - \gamma \in \mathbf{Z}$  and at  $\mu = \infty$  if  $\alpha - \beta \in \mathbf{Z}$ . Several sub-cases must be distinguished, and this is not the place to discuss them.

In our case  $\alpha = a, \beta = b + 1$  and  $\gamma = c + 1$ . Therefore  $\alpha - \beta = -2$ . This case is logarithmic. Two independent solutions are

$$\begin{aligned} \varphi_1^{(\infty)} &= \mu^{-\beta} g_1\left(\beta, \beta - \gamma + 1, 1 + \beta - \alpha; \frac{1}{\mu}\right) = (\text{in our case}) \mu^{-\alpha-2} g_1\left(\beta, \beta - \gamma + 1, 3; \frac{1}{\mu}\right), \\ \varphi_2^{(\infty)} &= \mu^{-\beta} F\left(\beta, \beta - \gamma + 1, 1 + \beta - \alpha; \frac{1}{\mu}\right) = (\text{in our case}) \mu^{-\beta} F\left(\beta, \beta - \gamma + 1, 3; \frac{1}{\mu}\right). \end{aligned}$$

Here,  $F$  is the Gauss hyper-geometric function and  $g_1$  is a logarithmic solution introduced in Norlund's paper [22], p 7:

$$\begin{aligned} g_1(u, v, w; z) &= \sum_{n=1}^{w-1} (-1)^{n-1} (n-1)! \frac{(u)_{-n} (v)_{-n}}{(w)_{-n}} \frac{1}{z^n} + F(u, v, w; z) \ln(-z) \\ &+ \sum_{n=0}^{\infty} \frac{(u)_n (v)_n}{n! (w)_n} [\psi(1 - u - n) + \psi(v + n) - \psi(w + n) - \psi(1 + n)] z^n, \end{aligned}$$

where

$$\begin{aligned} \psi(x) &:= \frac{d}{dx} \Gamma(x), \quad u \neq 1, 2, 3, \dots; \\ |z| &< 1, \quad \ln(-z) < 0 \quad \text{for } -1 < z < 0. \end{aligned}$$

Note that the first sum  $\sum_{n=1}^{w-1} (-1)^{n-1} (n-1)! \frac{(u)_{-n} (v)_{-n}}{(w)_{-n}} \frac{1}{z^n}$  is a polynomial in  $1/z$ .

If  $\theta_0, \theta_x \notin \mathbf{Z}$ , we fall in the non-logarithmic cases at  $\mu = 0, 1$ . Thus,

$$\begin{aligned} \varphi_1^{(0)} &= F\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} - \frac{1}{2}, \frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}, 1 + \theta_0; \mu\right), \\ \varphi_2^{(0)} &= \mu^{-\theta_0} F\left(-\frac{\theta_0}{2} + \frac{\theta_x}{2} - \frac{1}{2}, -\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}, 1 - \theta_0; \mu\right), \\ \varphi_1^{(1)} &= F\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} - \frac{1}{2}, \frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}, 1 + \theta_x; 1 - \mu\right), \\ \varphi_2^{(1)} &= (1 - \mu)^{-\theta_x} F\left(\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{1}{2}, \frac{\theta_0}{2} - \frac{\theta_x}{2} + \frac{3}{2}, 1 - \theta_x; 1 - \mu\right), \\ \varphi_1^{(\infty)} &= \mu^{-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{3}{2}} g_1\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}, -\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}, 3; \frac{1}{\mu}\right), \\ \varphi_2^{(\infty)} &= \mu^{-\frac{\theta_0}{2} - \frac{\theta_x}{2} - \frac{3}{2}} F\left(\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}, -\frac{\theta_0}{2} + \frac{\theta_x}{2} + \frac{3}{2}, 3; \frac{1}{\mu}\right). \end{aligned}$$

The connection matrix between  $[\varphi_1^{(0)}, \varphi_2^{(0)}]$  and  $[\varphi_1^{(1)}, \varphi_2^{(1)}]$  is a standard one:

$$[\varphi_1^{(0)}, \varphi_2^{(0)}] = [\varphi_1^{(1)}, \varphi_2^{(1)}] C_{01}, \quad 0 < \arg(1 - \mu) < 2\pi,$$

where  $C_{01}$  is (31).



On the other hand, the connection matrix between  $[\varphi_1^{(0)}, \varphi_2^{(0)}]$  and  $[\varphi_1^{(\infty)}, \varphi_2^{(\infty)}]$  is computed in [22]. Our specific case falls in the case  $\alpha - \beta = -p$ ,  $p \geq 0$  integer. From [22], p 27, case 11, formulae (1) and (2), we deduce the connection formulae

$$\begin{aligned} (-\mu)^{\alpha-\beta} F\left(\beta - \gamma + 1, \beta, 1 + \beta - \alpha; \frac{1}{\mu}\right) &= \frac{\Gamma(\beta - \alpha + 1)\Gamma(1 - \gamma)}{\Gamma(1 - \alpha)\Gamma(\beta - \gamma + 1)} (-\mu)^\alpha F(\alpha, \beta, \gamma; \mu) \\ &\quad + \frac{\Gamma(\beta - \alpha + 1)\Gamma(\gamma - 1)}{\Gamma(\gamma - \alpha)\Gamma(\beta)} (-\mu)^{\alpha-\gamma+1} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; \mu), \\ (-\mu)^{\alpha-\beta} g_1\left(\beta - \gamma + 1, \beta, 1 + \beta - \alpha; \frac{1}{\mu}\right) &= (-1)^{\alpha-\beta} \frac{\Gamma(1 + \beta - \alpha)\Gamma(1 - \beta)\Gamma(\alpha - \gamma + 1)}{\Gamma(2 - \gamma)} \\ &\quad \times (-\mu)^{\alpha-\gamma+1} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma; \mu). \end{aligned}$$

Here,  $|\arg(-\mu)| < \pi$ ,  $\ln(-\mu) > 0$  for  $-\infty < \mu < -1$ . The branch cut in the  $\mu$ -plane is  $[0, +\infty)$ . As for the minus signs, we choose  $-\mu = e^{-i\pi}\mu$ , then  $0 < \arg \mu < 2\pi$ . Moreover, in our case  $(-1)^{\alpha-\beta} = 1$ . With this preparation, we can write the connection matrix:

$$[\varphi_1^{(\infty)}, \varphi_2^{(\infty)}] = [\varphi_1^{(0)}, \varphi_2^{(0)}] C_{\infty 0}, \quad 0 < \arg \mu < 2\pi,$$

where  $C_{\infty 0}$  is (32).

In order to write  $\Psi_{\text{IN}}^{\text{Match}}$  in terms of the  $\varphi_i^{(\infty)}$ , let us compute the behaviour for  $\mu \rightarrow \infty$  of

$$\mu^{\theta_0/2} (\mu - 1)^{\theta_x/2} \begin{pmatrix} \varphi_1^{(\infty)} & \varphi_2^{(\infty)} \\ \xi_1^{(\infty)} & \xi_2^{(\infty)} \end{pmatrix}.$$

We have

$$\begin{aligned} \mu^{\frac{\theta_0}{2}} (\mu - 1)^{\frac{\theta_x}{2}} \varphi_1^{(\infty)} &= -\frac{32}{[(\theta_0 + \theta_x)^2 - 1][(\theta_0 - \theta_x)^2 - 1]} \\ &\quad \times \mu^{\frac{1}{2}} \left[ 1 + O\left(\frac{1}{\mu}\right) \right] + \mu^{-\frac{3}{2}} \ln \mu \left[ 1 + O\left(\frac{1}{\mu}\right) \right]. \end{aligned}$$

$$\mu^{\frac{\theta_0}{2}} (\mu - 1)^{\frac{\theta_x}{2}} \varphi_2^{(\infty)} = \mu^{-\frac{3}{2}} \left[ 1 + O\left(\frac{1}{\mu}\right) \right].$$

$$\mu^{\frac{\theta_0}{2}} (\mu - 1)^{\frac{\theta_x}{2}} \xi_1^{(\infty)} = -\frac{2}{r} \mu^{-\frac{1}{2}} \ln \mu \left[ 1 + O\left(\frac{1}{\mu}\right) \right] + O(\mu^{-\frac{1}{2}}),$$

$$\mu^{\frac{\theta_0}{2}} (\mu - 1)^{\frac{\theta_x}{2}} \xi_2^{(\infty)} = \frac{2}{r} \mu^{-\frac{1}{2}} \left[ 1 + O\left(\frac{1}{\mu}\right) \right].$$

Therefore,

$$\mu^{\frac{\theta_0}{2}} (\mu - 1)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(\infty)} & \varphi_2^{(\infty)} \\ \xi_1^{(\infty)} & \xi_2^{(\infty)} \end{pmatrix} = \left[ I + O\left(\frac{1}{\mu}\right) \right] \mu^{\frac{1}{2}\sigma_3} \begin{pmatrix} 1 & 0 \\ R \ln \mu & 1 \end{pmatrix} \begin{pmatrix} -\frac{2}{rR} & 0 \\ 0 & \frac{2}{r} \end{pmatrix}.$$

We conclude that the matrix  $\Psi_0$  used in the  $\Psi_{\text{OUT}} \leftrightarrow \Psi_{\text{IN}}$  matching is

$$\begin{aligned} \Psi_0 &= \mu^{\frac{\theta_0}{2}} (\mu - 1)^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(\infty)} & \varphi_2^{(\infty)} \\ \xi_1^{(\infty)} & \xi_2^{(\infty)} \end{pmatrix} C, \\ C &:= \begin{pmatrix} -R\frac{r}{2} & 0 \\ 0 & \frac{r}{2} \end{pmatrix} = \begin{pmatrix} \frac{[1-(\theta_0+\theta_x)^2][(\theta_0-\theta_x)^2-1]}{32} & 0 \\ 0 & \frac{r}{2} \end{pmatrix}. \end{aligned}$$

As we have already remarked, we do the transformation  $\Psi_{\text{IN}}^{\text{Match}} \mapsto \Psi_{\text{IN}}^{\text{Match}} C_{\text{IN}}^{-1}$ , to compute the  $x$ -independent monodromy at  $\lambda = 0, x$ . This means that we have to compute the

monodromy of

$$\begin{aligned} \Psi_{\text{IN}}^{\text{Match}} C_{\text{IN}}^{-1} &= K_0(x) \Psi_0(\lambda/x) = K_0(x) \begin{pmatrix} \lambda \\ x \end{pmatrix}^{\frac{\theta_0}{2}} \begin{pmatrix} \lambda \\ x-1 \end{pmatrix}^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(\infty)} & \varphi_2^{(\infty)} \\ \xi_1^{(\infty)} & \xi_2^{(\infty)} \end{pmatrix} C, \\ &= K_0(x) \begin{pmatrix} \lambda \\ x \end{pmatrix}^{\frac{\theta_0}{2}} \begin{pmatrix} \lambda \\ x-1 \end{pmatrix}^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(0)} & \varphi_2^{(0)} \\ \xi_1^{(0)} & \xi_2^{(0)} \end{pmatrix} C_{\infty 0} C, \\ &= K_0(x) \begin{pmatrix} \lambda \\ x \end{pmatrix}^{\frac{\theta_0}{2}} \begin{pmatrix} \lambda \\ x-1 \end{pmatrix}^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} \\ \xi_1^{(1)} & \xi_2^{(1)} \end{pmatrix} C_{01} C_{\infty 0} C. \end{aligned}$$

In order to do this, we observe that from the definition of  $\varphi_1^{(0)}, \varphi_2^{(0)}$  it follows that

$$K_0(x) \begin{pmatrix} \lambda \\ x \end{pmatrix}^{\frac{\theta_0}{2}} \begin{pmatrix} \lambda \\ x-1 \end{pmatrix}^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(0)} & \varphi_2^{(0)} \\ \xi_1^{(0)} & \xi_2^{(0)} \end{pmatrix} = \psi_0^{\text{IN}}(x) (I + O(\lambda)) \lambda^{\frac{\theta_0}{2} \sigma_3},$$

for  $\lambda \rightarrow 0$ . From the definition of  $\varphi_1^{(1)}, \varphi_2^{(1)}$ , it follows that

$$K_0(x) \begin{pmatrix} \lambda \\ x \end{pmatrix}^{\frac{\theta_0}{2}} \begin{pmatrix} \lambda \\ x-1 \end{pmatrix}^{\frac{\theta_x}{2}} \begin{pmatrix} \varphi_1^{(1)} & \varphi_2^{(1)} \\ \xi_1^{(1)} & \xi_2^{(1)} \end{pmatrix} = \psi_x^{\text{IN}}(x) (I + O(\lambda - x)) (\lambda - x)^{\frac{\theta_x}{2} \sigma_3},$$

for  $\lambda \rightarrow x$ . It is not necessary here to explicitly write the invertible matrices  $\psi_0^{\text{IN}}(x)$  and  $\psi_x^{\text{IN}}(x)$ . The above construction implies that  $\Psi_{\text{IN}}^{\text{Match}} C_{\text{IN}}^{-1}$  has monodromy matrices at  $\lambda = 0, x$ , respectively, given by

$$\begin{aligned} M_0 &= C^{-1} (C_{\infty 0})^{-1} \exp\{i\pi \theta_0 \sigma_3\} C_{\infty 0} C, \\ M_x &= C^{-1} (C_{\infty 0})^{-1} (C_{01})^{-1} \exp\{i\pi \theta_x \sigma_3\} C_{01} C_{\infty 0} C. \end{aligned}$$

These matrices coincide with the monodromy matrices of  $\Psi C_{\text{IN}}^{-1}$ .

As a last simplification, we consider the transformation  $M \mapsto CMC^{-1}$ . We obtain the result of theorem 3, case (c). Namely,

$$\begin{aligned} M_0 &= (C_{\infty 0})^{-1} \exp\{i\pi \theta_0 \sigma_3\} C_{\infty 0}, & M_{\infty} &= \begin{pmatrix} -1 & 0 \\ 2\pi i(1 - \frac{r_1}{\rho}) & -1 \end{pmatrix}, \\ M_1 &= \begin{pmatrix} 1 & 0 \\ 2\pi i \frac{r_1}{\rho} & 1 \end{pmatrix}, & M_x &= (C_{\infty 0})^{-1} (C_{01})^{-1} \exp\{i\pi \theta_x \sigma_3\} C_{01} C_{\infty 0}. \end{aligned}$$

We observe that  $r$  does not appear in the monodromy matrices.

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### Appendix A

**Proposition 2.** Let  $B_0, B_1$  be  $2 \times 2$  matrices such that

$$\text{eigenvalues}(B_0) = 0, -c, \quad \text{eigenvalues}(B_1) = 0, c - a - b,$$

and  $B_0 + B_1$  is either diagonalizable:

$$B_0 + B_1 = \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix} \quad (\text{it may happen that } a = b)$$

or it is a Jordan form:

$$B_0 + B_1 = \begin{pmatrix} -a & 1 \\ 0 & -a \end{pmatrix}.$$

Then,  $B_0$  and  $B_1$  can be computed as in the following cases. Let  $r, s$  be any complex numbers.

(1) Diagonalizable case

Case  $a \neq b$

$$B_0 := \begin{pmatrix} \frac{a(b-c)}{a-b} & r \\ \frac{ab(a-c)(c-b)}{r(a-b)^2} & \frac{b(c-a)}{a-b} \end{pmatrix}, \quad B_1 = \begin{pmatrix} \frac{a(c-a)}{a-b} & -r \\ -(B_0)_{21} & \frac{b(b-c)}{a-b} \end{pmatrix}, \quad r \neq 0 \quad (\text{A.1})$$

$$\text{if } a = 0: \quad B_0 = \begin{pmatrix} 0 & r \\ 0 & -c \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & -r \\ 0 & c-b \end{pmatrix}, \quad (\text{A.2})$$

$$\text{if } b = 0: \quad B_0 = \begin{pmatrix} -c & r \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} c-a & -r \\ 0 & 0 \end{pmatrix}, \quad (\text{A.3})$$

$$\text{if } a = c \neq b: \quad B_0 = \begin{pmatrix} -a & r \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & -r \\ 0 & -b \end{pmatrix}, \quad (\text{A.4})$$

$$\text{if } b = c \neq a: \quad B_0 = \begin{pmatrix} 0 & r \\ 0 & -b \end{pmatrix}, \quad B_1 = \begin{pmatrix} -a & -r \\ 0 & 0 \end{pmatrix}. \quad (\text{A.5})$$

Cases (A.2)–(A.5) are already included in (A.1).

Case  $a = b$ . We have two sub-cases:

$$\text{if } a = b = c: \quad B_0 = \begin{pmatrix} -c-s & r \\ -\frac{s(c+s)}{r} & s \end{pmatrix}, \quad B_1 = \begin{pmatrix} s & -r \\ \frac{s(c+s)}{r} & -c-s \end{pmatrix}, \quad (\text{A.6})$$

$$\text{if } a = b = 0: \quad B_0 = \begin{pmatrix} -c-s & r \\ -\frac{s(c+s)}{r} & s \end{pmatrix}, \quad B_1 = -B_0. \quad (\text{A.7})$$

The transpose matrices of all the above cases are also possible.

(2) Jordan case

For  $a \neq 0$  and  $a \neq c$ , we have

$$B_0 = \begin{pmatrix} r & \frac{r(r+c)}{a(a-c)} \\ a(c-a) & -c-r \end{pmatrix}, \quad B_1 = \begin{pmatrix} -a-r & 1 - \frac{r(r+c)}{a(a-c)} \\ a(a-c) & c-a+r \end{pmatrix}. \quad (\text{A.8})$$

For  $a = 0$ , or  $a = c$ , we have two possibilities:

$$B_0 = \begin{pmatrix} 0 & r \\ 0 & -c \end{pmatrix}, \quad B_1 = \begin{pmatrix} -a & 1-r \\ 0 & -a+c \end{pmatrix}; \quad (\text{A.9})$$

or

$$B_0 = \begin{pmatrix} -c & r \\ 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} c-a & 1-r \\ 0 & -a \end{pmatrix} \quad (\text{A.10})$$

**Proposition 3.** Let  $B_0$  and  $B_1$  be as in proposition 2. The linear system

$$\frac{d}{dz} \begin{pmatrix} \varphi \\ \xi \end{pmatrix} = \left[ \frac{B_0}{z} + \frac{B_1}{z-1} \right] \begin{pmatrix} \varphi \\ \xi \end{pmatrix}.$$

may be reduced to a Gauss hyper-geometric equation, in the following cases. Diagonalizable case (i.e., from (A.1) to (A.7)):

$$z(1-z)\frac{d^2\varphi}{dz^2} + (1+c - (a + [b+1] + 1)z)\frac{d\varphi}{dz} - a(b+1)\varphi = 0. \tag{A.11}$$

The component  $\xi$  is obtained by the following equalities, according to the different cases of proposition 2. Cases (A.1)–(A.5):

$$\xi = \frac{1}{r} \left[ z(1-z)\frac{d\varphi}{dz} - a \left( z + \frac{b-c}{a-b} \right) \varphi \right]. \tag{A.12}$$

Case (A.6):

$$\xi = \frac{1}{r} \left[ z(1-z)\frac{d\varphi}{dz} + (c+s - cz)\varphi \right].$$

Case (A.7):

$$\xi = \frac{1}{r} \left[ z(1-z)\frac{d\varphi}{dz} + (c+s)\varphi \right].$$

Jordan case (A.8). The equation for  $\varphi$  is in Gauss hyper-geometric form only when  $r = -a$ . In this case, the matrices (A.8) are

$$B_0 = \begin{pmatrix} -a & 1 \\ a(c-a) & a-c \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & 0 \\ a(a-c) & c-2a \end{pmatrix}.$$

The equation is

$$z(1-z)\frac{d^2\varphi}{dz^2} + [1+c - (2a+1)z]\frac{d\varphi}{dz} - a^2\varphi = 0, \quad \xi = z\frac{d\varphi}{dz} + a\varphi.$$

Jordan case (A.9). For  $r = 1$  we get

$$\text{for } a = 0, \quad z(1-z)\frac{d^2\varphi}{dz^2} + (1+c-z)\frac{d\varphi}{dz} = 0,$$

$$\text{for } c = a, \quad z(1-z)\frac{d^2\varphi}{dz^2} + (1+a - (2a+1)z)\frac{d\varphi}{dz} + \left(-a^2 - \frac{a}{1-z}\right)\varphi = 0.$$

The Gauss form appears only when  $a = 0$ .

Jordan case (A.10). For  $r = 1$  we get

$$\text{for } a = 0, \quad z(1-z)\frac{d^2\varphi}{dz^2} + (1+c-z)\frac{d\varphi}{dz} + \frac{c}{1-z}\varphi = 0,$$

$$\text{for } c = a, \quad z(1-z)\frac{d^2\varphi}{dz^2} + (1+a - (2a+1)z)\frac{d\varphi}{dz} - a^2\varphi = 0.$$

For  $r \neq 1$ , we do not get a Gauss hyper-geometric form for the equation of  $\varphi$  in both cases (A.9) and (A.10). Nevertheless, the matrices are in upper triangular form, so the equation for  $\xi$  is solvable by elementary integration.

As a compendium to the above proposition, we recall that any irreducible representation of  $\pi_1(\mathbf{CP}^1 \setminus \{0, 1, \infty\}) \mapsto GL(2, \mathbf{C})$  can be realized as the monodromy group of a Riemann (or Gauss) equation. A reducible representation (namely,  $M_0, M_1, M_\infty$  are in upper triangular form) can be realized by the monodromy of a  $2 \times 2$  Fuchsian system

$$\frac{dY}{dz} = \left[ \frac{B_0}{z} + \frac{B_1}{z-1} \right] Y,$$

where  $B_0, B_1$  are  $2 \times 2$  upper triangular matrices. We also state the following:

**Lemma 12.** Consider a  $2 \times 2$  linear system

$$\frac{dY(z)}{dz} = A(z)Y, \quad A(z) = \begin{pmatrix} a(z) & b(z) \\ 0 & c(z) \end{pmatrix}$$

such that  $A(z)$  is meromorphic, with poles  $a_1, a_2, \dots, a_N, \infty$ . The monodromy group is generated by  $N$  upper triangular monodromy matrices:

$$M_i = \begin{pmatrix} \lambda_1^{(i)} & R^{(i)} \\ 0 & \lambda_2^{(i)} \end{pmatrix}, \quad i = 1, 2, \dots, N,$$

where  $\lambda_1^{(i)}, \lambda_2^{(i)}, R^{(i)}$  are constants (i.e., they are independent of  $z$ ) given by

$$\lambda_1^{(i)} = \exp\{2\pi i \operatorname{Res} a(z)|_{a_i}\}, \quad \lambda_2^{(i)} = \exp\{2\pi i \operatorname{Res} c(z)|_{a_i}\},$$

$$R^{(i)} = \int_z^{z \exp\{2\pi i\}} ds b(s) \frac{u_2(s)}{u_1(s)} + (\lambda_1^{(i)} - \lambda_2^{(i)}) \int_{z_0}^z ds b(s) \frac{u_2(s)}{u_1(s)}.$$

$R^{(i)}$  depends on a chosen non-singular base point  $z_0$ , but not on  $z$ . One of the non-zero  $R^{(i)}$  can be put equal to 1, by redefining

$$Y \mapsto YC, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1/R^{(i)} \end{pmatrix}.$$

**Proof.** Let us write  $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$  and the equation in the form

$$\frac{dy_1}{dz} = a(z)y_1 + b(z)y_2, \quad \frac{dy_2}{dz} = c(z)y_2.$$

Let  $z_0 \neq a_i, \infty, i = 1, \dots, N$ . The second equation has solution  $y_2(z) = C_2 u_2$ , where  $u_2 := \exp\{\int_{z_0}^z ds c(s)\}$ ,  $C_2 \in \mathbf{C}$ . The first equation becomes

$$\frac{dy_1}{dz} = a(z)y_1 + C_2 b(z)u_2(z). \quad (\text{A.13})$$

We solve the first equation by variation of parameters. Let  $u_1(z) = \exp\{\int_{z_0}^z ds a(s)\}$  be a fundamental solution of the homogeneous equation  $\frac{dy_1}{dz} = a(z)y_1$ . We look for a solution of (A.13) of the form  $y_2(z) = w(z)u_1(z)$ . Substitution gives

$$\frac{dw}{dz} = C_2 b(z) \frac{u_2(z)}{u_1(z)} \implies w(z) = C_2 v(z) + C_1, \quad C_1 \in \mathbf{C},$$

where

$$v(z) = \int_{z_0}^z ds b(s) \frac{u_2(s)}{u_1(s)}.$$

The general solution of (A.13) is  $y_1(z) = C_1 u_1(z) + C_2 v(z)u_1(z)$ . Then, a fundamental solution for the initial system can be chosen to be

$$Y(z) = \begin{pmatrix} u_1(z) & v(z)u_1(z) \\ 0 & u_2(z) \end{pmatrix}.$$

We compute the monodromy for  $(z - a_i) \mapsto (z - a_i) e^{2\pi i}$ . We have

$$\begin{aligned} u_1(z) &\mapsto \lambda_1^{(i)} u_1(z), & \lambda_1^{(i)} &= \exp\{2\pi i \operatorname{Res} a(z)|_{a_i}\}, \\ u_2(z) &\mapsto \lambda_2^{(i)} u_2(z), & \lambda_2^{(i)} &= \exp\{2\pi i \operatorname{Res} c(z)|_{a_i}\}. \end{aligned} \quad (\text{A.14})$$

By linearity, the vector solution  $\begin{pmatrix} v(z)u_1(z) \\ u_2(z) \end{pmatrix}$  is transformed into a linear combination of two independent vector solutions:

$$R^{(i)} \begin{pmatrix} u_1(z) \\ 0 \end{pmatrix} + S^{(i)} \begin{pmatrix} v(z)u_1(z) \\ u_2(z) \end{pmatrix}.$$

Moreover,  $S^{(i)}$  must coincide with  $\lambda_2^{(i)}$ , because—by virtue of (A.14)— $u_2(z) \mapsto \lambda_2^{(i)} u_2(z)$ . Namely,

$$\begin{pmatrix} v(z)u_1(z) \\ u_2(z) \end{pmatrix} \mapsto R^{(i)} \begin{pmatrix} u_1(z) \\ 0 \end{pmatrix} + \lambda_2^{(i)} \begin{pmatrix} v(z)u_1(z) \\ u_2(z) \end{pmatrix}, \quad R^{(i)} \in \mathbf{C}.$$

Thus,

$$\begin{aligned} Y(z) &= \begin{pmatrix} u_1(z) & v(z)u_1(z) \\ 0 & u_2(z) \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1^{(i)} u_1(z) & \lambda_2^{(i)} v(z)u_1(z) + R^{(i)} u_1(z) \\ 0 & \lambda_2^{(i)} u_2(z) \end{pmatrix} \\ &= Y(z) \begin{pmatrix} \lambda_1^{(i)} & R^{(i)} \\ 0 & \lambda_2^{(i)} \end{pmatrix}. \end{aligned}$$

Let  $C_i$  be a small loop around  $a_i$ . To find  $R^{(i)}$ , let us observe that  $u_1(z)v(z) \mapsto \lambda_1^{(i)} u_1(z)\tilde{v}(z)$ , where  $\tilde{v}(z) = v(z) + K_i(z)$ ,  $K_i(z) := \int_z^{z \exp\{2\pi i\}} ds b(s) \frac{u_2(s)}{u_1(s)}$ . Thus,

$$Y(z) = \begin{pmatrix} u_1(z) & v(z)u_1(z) \\ 0 & u_2(z) \end{pmatrix} \mapsto \begin{pmatrix} \lambda_1^{(i)} u_1(z) & \lambda_1^{(i)} v(z)u_1(z) + K_i(z)u_1(z) \\ 0 & \lambda_2^{(i)} u_2(z) \end{pmatrix}.$$

We must have  $\lambda_1^{(i)} v(z)u_1(z) + K_i(z)u_1(z) \equiv \lambda_2^{(i)} v(z)u_1(z) + R^{(i)} u_1(z)$ ; namely,

$$R^{(i)} = K_i(z) + (\lambda_1^{(i)} - \lambda_2^{(i)})v(z). \quad \square$$

### Appendix B. Formal asymptotic expansion

(1) We consider systems (59):

$$\frac{dY}{dz} = \left[ \Omega + \frac{D_1}{z} + \sum_{n=2}^{\infty} \frac{D_n}{z^n} \right] Y := D(z)Y, \quad \Omega = \text{diag}(\omega_1, \omega_2, \dots, \omega_n),$$

with  $\omega_i \neq \omega_j$ , for  $i \neq j$ . We introduce a gauge transformation  $Y = G(z)\tilde{Y}$ , such that

$$\frac{d\tilde{Y}}{dz} = \left[ G^{-1}(z)D(z)G(z) - G^{-1}(z)\frac{dG(z)}{dz} \right] \tilde{Y}$$

be the simple form

$$\frac{d\tilde{Y}}{dz} = \left[ \Omega + \frac{\Omega_1}{z} \right] \tilde{Y}, \quad \Omega, \Omega_1 \text{ diagonal.}$$

Formally, we write  $G(z)$  as

$$G(z) = I + \frac{G_1}{z} + \frac{G_2}{z^2} + \dots, \quad z \rightarrow \infty.$$

If  $G_n$ 's can be determined, we get the formal solution

$$Y(z) \sim \left[ I + \sum_{n=1}^{\infty} \frac{G_n}{z^n} \right] \exp\{z\Omega + \Omega_1 \ln(z)\}, \quad z \rightarrow \infty.$$

For a sector of angular with  $\pi + \epsilon$ ,  $\epsilon > 0$ , sufficiently small (but finite and non-zero), there exists a unique solution  $Y(z)$  with the above asymptotic expansion [3].

In order to determine  $G_n$  and  $\Omega_1$ , we solve  $D(z)G(z) - \partial_z G(z) = G(z)(\Omega_0 + \Omega_1 z^{-1})$ :

$$\left(\Omega + \sum_{n=1}^{\infty} \frac{D_n}{z^n}\right) \left(I + \sum_{n=1}^{\infty} \frac{G_n}{z^n}\right) + \sum_{n=2}^{\infty} \frac{(n-1)G_{n-1}}{z^n} = \left(I + \sum_{n=1}^{\infty} \frac{G_n}{z^n}\right) \left(\Omega + \frac{\Omega_1}{z}\right).$$

We identify equal powers of  $z^{-1}$ . From the power  $1/z$  we get

$$\Omega_1 = \text{diag}(\omega_1^{(1)}, \dots, \omega_n^{(1)}), \quad \omega_i^{(1)} := (\Omega_1)_{ii} = (D_1)_{ii}, \quad (G_1)_{ij} = -\frac{(D_1)_{ij}}{\omega_i - \omega_j}.$$

From the power  $1/z^2$ , for  $i \neq j$ , we compute  $(G_2)_{ij}$ , and for  $i = j$  we compute

$$(G_1)_{ii} = -(D_2)_{ii} - \sum_{k \neq i} (D_1)_{ik} (G_1)_{ki}.$$

From the power  $1/z^n$  we get

$$(G_{n-1})_{ii} = \frac{1}{n-1} \left\{ -(D_n + D_{n-1}G_1 + \dots + D_2G_{n-2})_{ii} - \sum_{k \neq i} (D_1)_{ik} (G_{n-1})_{ki} \right\},$$

$$(G_n)_{ij} = \frac{1}{\omega_i - \omega_j} \left\{ [\omega_j^{(1)} - \omega_i^{(1)} - (n-1)] (G_{n-1})_{ij} - \sum_{k \neq i} (D_1)_{ik} (G_{n-1})_{kj} \right. \\ \left. - (D_n + D_{n-1}G_1 + \dots + D_2G_{n-2})_{ij} \right\}, \quad i \neq j$$

(2) We consider system (60):

$$\frac{dY}{dz} = \left[ x^2 \Lambda z + x \Lambda + \frac{E_1}{z} + \sum_{n=2}^{\infty} \frac{E_n}{z^n} \right] Y := E(z)Y, \quad \Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n),$$

with  $\lambda_i \neq \lambda_j$ , for  $i \neq j$ . We introduce a gauge transformation

$$Y(z) = K(z) \tilde{Y}(z)$$

in order to reach the simple form

$$\frac{d\tilde{Y}}{dz} = \left[ K^{-1}(z)E(z)K(z) - K^{-1}(z) \frac{dK(z)}{dz} \right] \tilde{Y} \\ \equiv \left[ x^2 \Lambda z + x \Lambda + \frac{\Lambda_1}{z} \right] \tilde{Y}, \quad \Lambda_1 \text{ diagonal.}$$

Formally, we write

$$K(z) \sim I + \frac{K_1}{z} + \frac{K_2}{z^2} + \dots = I + \sum_{n=1}^{\infty} \frac{K_n}{z^n}.$$

Provided that we can determine the matrices  $K_n$ , we obtain the formal solution

$$Y(z) \sim \left[ I + \sum_{n=1}^{\infty} \frac{K_n}{z^n} \right] \exp \left\{ \frac{x^2}{2} \Lambda z^2 + x \Lambda z \Lambda_1 \ln x \right\}.$$

For a sector of angular with  $\frac{\pi}{2} + \epsilon$ ,  $\epsilon > 0$ , sufficiently small (but finite and non-zero), there exists a unique solution  $Y(z)$  with the above asymptotic expansion [3].

In order to determine  $K_n$  and  $\Lambda_1$ , we solve  $E(z)K(z) - \partial_z K(z) = K(z)(x^2\Lambda z^2 + \Lambda z + \Omega_1 z^{-1})$ :

$$\begin{aligned} & \left(x^2\Lambda z^2 + \Lambda z + \sum_{n=1}^{\infty} \frac{E_n}{z^n}\right) \left(I + \sum_{n=1}^{\infty} \frac{K_n}{z^n}\right) + \sum_{n=2}^{\infty} \frac{(n-1)K_{n-1}}{z^n} \\ &= \left(I + \sum_{n=1}^{\infty} \frac{K_n}{z^n}\right) \left(x^2\Lambda z^2 + \Lambda z + \frac{\Omega_1}{z}\right). \end{aligned}$$

We identify equal powers of  $z^{-1}$ .

*Power  $z$ .* It is an identity:  $x^2\Lambda = x^2\Lambda$ .

*Power  $z^0$ :*

$$x\Lambda + x^2\Lambda K_1 = x\Lambda + x^2K_1\Lambda \implies [\Lambda, K_1] = 0.$$

This means that  $K_1$  is a diagonal matrix.

*Power  $1/z$ .* We obtain the equation

$$x^2[\Lambda, K_2] = \Lambda_1 - E_1 \implies (\Lambda_1)_{ii} = (E_1)_{ii}, \quad (G_2)_{ij} = -\frac{(E_1)_{ij}}{x^2(\lambda_i - \lambda_j)}, \quad i \neq j.$$

*Power  $1/z^2$ .* We obtain the equation

$$x^2[\Lambda, K_3] = x[K_2, \Lambda] + K_1\Lambda_1 - E_1K_1 - K_1 - E_2 \equiv \frac{E_1 - \Lambda_1}{x} + K_1\Lambda_1 - E_1K_1 - K_1 - E_2.$$

Thus,

$$(K_1)_{ii} = -(E_2)_{ii} \quad (K_3)_{ij} = \frac{1}{x^2(\lambda_i - \lambda_j)} \left[ (E_1)_{ij} \left( \frac{1}{x} + (E_2)_{jj} \right) - (E_2)_{ij} \right], \quad i \neq j.$$

*Power  $1/z^2$ .* We obtain the equation

$$x^2[\Lambda, K_4] = x[K_3, \Lambda] + K_2\Lambda_1 - E_1K_2 - E_2K_1 - 2K_2 - E_3.$$

The diagonal part gives

$$2(K_2)_{ij} = [(E_2)_{ii}]^2 - (E_3)_{ii} - \sum_{k \neq i} (E_1)_{ik}(K_2)_{ki}.$$

The non-diagonal part gives  $(K_4)_{ij}, i \neq j$ .

We content ourselves with these results, namely, the determination of  $K_1$  and  $K_2$ . With the same procedure, we can determine all  $K_n$ 's.

### Appendix C. Birational transformations

All the solutions of (PVI) of the form

$$y(x) = b_0 + b_1x + b_2x^2 + \dots + b_Nx^N + \dots \tag{C.1}$$

are obtained from the matching procedure of sections 5–7. By this we mean that solutions of type (C.1) are the solutions given by the matching procedure or they can be obtained from solutions given by the matching procedure via one of the birational transformations of [23] and the transformation (19).

Birational transformations are symmetries of (PVI), namely invertible transformations

$$y'(x) = \frac{P(x, y(x))}{Q(x, y(x))}, \quad x' = \frac{p(x)}{q(x)}, \quad (\theta_0, \theta_x, \theta_1, \theta_\infty) \mapsto (\theta'_0, \theta'_x, \theta'_1, \theta'_\infty)$$



such that  $y(x)$  satisfies (PVI) with coefficients  $\theta_0, \theta_x, \theta_1, \theta_\infty$  and variable  $x$ , if and only if  $y'(x')$  satisfies (PVI) with coefficients  $\theta'_0, \theta'_x, \theta'_1, \theta'_\infty$  and variable  $x'$ . The functions  $P, Q$  are polynomials,  $p, q$  are linear, the transformation of  $\theta_\mu$ 's is an element of a linear representation of one of the following groups—permutation group, the Weyl group of the root system  $D_4$ , the group of translations  $v := (v_1, v_2, v_3, v_4) \mapsto v + e_j, j = 1, 2, 3, 4$  (where  $e_1 = (1, 0, 0, 0), \dots, e_4 = (0, 0, 0, 1)$ ):<sup>7</sup>

\* *Permutation group*

$$\begin{aligned} x^1: & \theta'_1 = \theta_0, \quad \theta'_0 = \theta_1; \quad \theta'_x = \theta_x, \quad \theta'_\infty = \theta_\infty; \quad y'(x) = y(x) - 1, \quad x = 1 - x'. \\ x^2: & \theta'_0 = \theta_\infty - 1, \quad \theta'_\infty = \theta_0 + 1; \quad \theta'_1 = \theta_1, \quad \theta'_x = \theta_x; \quad y'(x) = \frac{1}{y(x)}, \quad x = \frac{1}{x'}. \\ x^3: & \theta'_0 = \theta_x, \quad \theta'_x = \theta_0; \quad \theta'_1 = \theta_1, \quad \theta'_\infty = \theta_\infty; \quad y'(x) = \frac{x - y(x)}{x - 1}, \quad x = \frac{x'}{x' - 1}. \end{aligned}$$

\* *Weyl group*

$$\begin{aligned} w_1: & \theta'_1 = -\theta_1; \quad \theta'_0 = \theta_0, \quad \theta'_x = \theta_x, \quad \theta'_\infty = \theta_\infty. \\ w_2: & \theta'_0 = \frac{\theta_0 + \theta_1 + \theta_x + \theta_\infty}{2} - 1, \quad \theta'_1 = \frac{\theta_0 + \theta_1 - \theta_x - \theta_\infty}{2} + 1, \\ & \theta'_x = \frac{\theta_0 - \theta_1 + \theta_x - \theta_\infty}{2} + 1, \quad \theta'_\infty = \frac{\theta_0 - \theta_1 - \theta_x + \theta_\infty}{2} + 1. \\ w_3: & \theta'_\infty = 2 - \theta_\infty; \quad \theta'_0 = \theta_0, \quad \theta'_x = \theta_x, \quad \theta'_1 = \theta_1. \\ w_4: & \theta'_\infty = 2 - \theta_\infty; \quad \theta'_x = 2 - \theta_x; \quad \theta'_0 = \theta_0, \quad \theta'_1 = \theta_1. \end{aligned}$$

The variable  $x' = x$ , but  $y'(x)$  is quite complicated and will not be given here (see [23]).

\* *Shift  $l_j : v \mapsto v + e_j$*

$$\begin{aligned} l_1: & \theta'_0 = \theta_0 + 1, \quad \theta'_1 = \theta_1 + 1; \quad \theta'_x = \theta_x, \quad \theta'_\infty = \theta_\infty; \\ l_2: & \theta'_0 = \theta_0 + 1, \quad \theta'_1 = \theta_1 - 1; \quad \theta'_x = \theta_x, \quad \theta'_\infty = \theta_\infty; \\ l_3: & \theta'_x = \theta_x + 1, \quad \theta'_\infty = \theta_\infty + 1; \quad \theta'_0 = \theta_0, \quad \theta'_1 = \theta_1; \\ l_4: & \theta'_x = \theta_x + 1, \quad \theta'_\infty = \theta_\infty - 1; \quad \theta'_0 = \theta_0, \quad \theta'_1 = \theta_1. \end{aligned}$$

The variable  $x' = x$ , but  $y'(x)$  is quite complicated and will not be given here (see [23]).

For the Taylor solutions, we have  $\sigma = \theta_1 - \theta_\infty$ . Denote  $\sigma' := \theta'_1 - \theta'_\infty$ . If we start from a Taylor solution constructed in sections 5–7 by means of the matching procedure, developed for  $\sigma = \theta_1 - \theta_\infty$ , then the birational transformations allow us to obtain the solutions defined for  $\sigma = \pm(\theta_1 \pm \theta_\infty) + n, n \in \mathbf{Z}$ . This is a consequence of the following actions:

$$\begin{aligned} l_1 \text{ and } l_4: & \quad \sigma \mapsto \sigma' = \sigma + 1; \quad l_2 \text{ and } l_3: \quad \sigma \mapsto \sigma' = \sigma - 1; \\ w_3 \cdot l_1 \cdot l_1: & \quad \sigma \mapsto \sigma' = \theta_1 + \theta_\infty; \quad w_1 \cdot w_3 \cdot l_1 \cdot l_1: \sigma \mapsto \sigma' = -\sigma. \end{aligned}$$

Note also that (PVI) is invariant for  $\theta_1 \mapsto -\theta_1$ . This maps  $\sigma \mapsto -\sigma$ . Other actions are

$$\begin{aligned} w_1: & \quad \sigma' = -(\theta_1 + \theta_\infty); \\ w_2: & \quad \sigma' = \sigma; \\ w_3: & \quad \sigma' = \theta_1 + \theta_\infty - 2; \\ w_4: & \quad \sigma' = \theta_1 + \theta_\infty - 2. \quad x^1: \quad \sigma' = \theta_0 - \theta_\infty; \\ & \quad x^2: \quad \sigma' = \theta_1 - \theta_0; \quad x^3: \quad \sigma' = \theta_1 - \theta_\infty = \sigma. \end{aligned}$$

<sup>7</sup> We note that  $\theta_1, \theta_x, \theta_0$  are defined up to the sign. Moreover,  $\theta_\infty$  is defined up to  $\theta_\infty \mapsto 2 - \theta_\infty$ , and for this reason symmetries are described in [23] in terms of  $\chi_\infty := \theta_\infty - 1$ .

**Appendix D. Examples of Taylor expansions**

We give the solutions of (PVI) of the form  $y(x) = \sum_{n=0}^{\infty} b_n x^n$ ,  $b_0 \neq 0$ , depending on the value of the coefficients  $\theta_0, \theta_1, \theta_x, \theta_{\infty}$ . Similar examples can be constructed for solutions  $y(x) = \sum_{n=1}^{\infty} b_n x^n$ ,  $b_1 \neq 0$ , by the symmetry (19).

We observe that, in general, the free parameter appears starting from some power  $x^m$ . If we truncate the series at  $x^{m-1}$ , we cannot see it.

We always denote by  $b_n$  the coefficients, though they change case by case.

**\* Example 1.** (PVI) always has one solution (15) when  $\theta_{\infty} - \theta_1 \notin \mathbf{Z}$  and one solution (18) when  $\theta_{\infty} + \theta_1 \notin \mathbf{Z}$ .

**\* Example 2.** Case  $\theta_1 + \theta_{\infty} = 0$ .

- (i) There are solutions (16), if  $\theta_x = \pm\theta_0$ ,  $\theta_{\infty} \neq 1$ .
- (ii) There is a solution (15), defined for  $\theta_{\infty} \neq 1, \frac{2n+1}{2}$ , where  $n \in \mathbf{Z}$ . The condition  $\theta_{\infty} \neq \frac{2n+1}{2}$  follows from the condition  $\theta_1 - \theta_{\infty} \neq n$  in (15), when  $\theta_1 = -\theta_{\infty}$ . The denominators of the coefficients  $b_n$  vanish for half-integer  $\theta_{\infty}$  and for  $\theta_{\infty} = 1$ .
- (iii) For  $\theta_{\infty} = \frac{2n+1}{2}$ , solutions  $y(x) = \sum_{n=0}^{\infty} b_n x^n$  in (ii) are not defined. On the other hand, solutions (16) are defined, provided that  $\theta_x = \pm\theta_0$ . Moreover, for a given  $\theta_{\infty} = \frac{2n+1}{2}$ , there may be solutions equivalent to (16), provided that  $\theta_x \pm \theta_0$  assumes some integer values. For example, consider  $\theta_{\infty} = \frac{3}{2}$ . We have (16):

$$y(x) = -2 + ax + \left(\theta_0^2 - 1 + \frac{3}{2}a - \frac{1}{2}a^3\right)x^2 + \sum_{n=3}^{\infty} b_n(a; \theta_0, \frac{3}{2})x^n, \quad \theta_x = \pm\theta_0;$$

and a solution equivalent to (16):

$$y(x) = 4 - (2 \pm \theta_0)x + ax^2 + \sum_{n=3}^{\infty} b_n(a; \theta_0)x^n, \quad \theta_x = \pm(1 + \theta_0), \quad \pm(1 - \theta_0).$$

- (iv) If  $\theta_{\infty} = 0, 1$ , we do not have any solution of the desired form, except for the singular solutions  $y = 0, 1$ . If  $\theta_{\infty} = 2$ , we have a solution equivalent to (17). The parameter  $a$  is the coefficient of  $x^3$  (coefficients of  $x^0, x$  and  $x^2$  have no parameter).

**\* Example 3.** Case  $\theta_{\infty} - \theta_1 = 0$ .

- (i) We have solutions (16).
- (ii) We have solution (18), with the substitution  $\theta_1 = \theta_{\infty}$ . This is defined for  $\theta_{\infty} \neq 1, \frac{2n+1}{2}$ .
- (iii) For  $\theta_{\infty} = \frac{2n+1}{2}$ , we find solutions of exactly the form (16), when  $\theta_x = \pm\theta_0$ . Moreover, for any given  $\theta_{\infty} = \frac{2n+1}{2}$ , there may be solutions equivalent to (16), provided that  $\theta_x \pm \theta_0$  has some integer value. For example, consider  $\theta_{\infty} = -\frac{1}{2}$ . There are solutions (16)

$$y(x) = \frac{2}{3} + ax + \sum_{n=2}^{\infty} b_n\left(a; \theta_0, -\frac{1}{2}\right)x^n, \quad \theta_x = \pm\theta_0;$$

and the equivalent solutions

$$y(x) = \frac{4}{3} - \frac{2 \pm \theta_0}{9}x + ax^2 + \sum_{n=3}^{\infty} b_n(a; \theta_0)x^n, \quad \theta_x = \pm(1 + \theta_0) \text{ or } \pm(1 - \theta_0).$$

Another example: consider  $\theta_{\infty} = \frac{1}{2}$ . In this case we just have (16) or the singular solution  $y = 1$ .

- (iv) For  $\theta_\infty = 0, 1$  we do not have solutions of the desired form, except for the singular solutions  $y = 0, 1$ . If  $\theta_\infty = 2$  we have a solution equivalent to (17). The parameter  $a$  is the coefficient of  $x^3$  (coefficients of  $x^0, x$  and  $x^2$  have no parameter).

**\* Example 4.** Case  $\theta_1 - \theta_\infty = -1$ .

- (i) We have solution (18), defined for  $\theta_\infty \neq 1, \frac{2n+1}{2}$ , where  $n \in \mathbf{Z}$ .  
 (ii) For any  $\theta_\infty = \frac{2n+1}{2}$ , solutions (16) are defined, when  $\theta_0 \pm \theta_x = 0$ . Moreover, solutions may exist equivalent to (16) by symmetry, provided that  $\theta_0 \pm \theta_x$  is some integer.  
 (iii) For  $\theta_\infty = 1$  we are exactly in case (17).  
 (iv) For  $\theta_\infty = 0, 2$  we have solutions equivalent to (17):

$$y(x) = 2 + \frac{\theta_0^2 - \theta_x^2 - 3}{6}x + ax^2 + \left\{ \frac{[(\theta_0 - \theta_x)^2 - 9][(\theta_0 + \theta_x)^2 - 9](\theta_0^2 - \theta_x^2)}{4320} + a \frac{\theta_0^2 - \theta_x^2 + 3}{6} \right\} x^3 + \dots$$

**\* Example 5.** Case  $\theta_1 - \theta_\infty = 1$ .

- (i) We have solution (18), with the substitution of  $\theta_1 = \theta_\infty + 1$ , and defined for  $\theta_\infty \neq 1, \frac{2n+1}{2}, n \in \mathbf{Z}$ .  
 (ii) Equivalent to (16) by symmetry, we have the solution

$$y(x) = \frac{2}{1 - \theta_\infty} + \frac{(\theta_\infty + 1)(2 \pm \theta_0)}{3(\theta_\infty - 1)}x + ax^2 + \sum_{n=3}^{\infty} b_n(a, \theta_\infty, \theta_0)x^n,$$

$$\theta_\infty \neq 0, 1; \quad \theta_x = \pm(1 - \theta_0) \quad \text{or} \quad \pm(1 + \theta_0).$$

The two signs in the coefficient of  $x$  depend on the choice  $\theta_x = \pm(1 - \theta_0)$  or  $\pm(1 + \theta_0)$ , respectively. Similar change of signs occur in all the coefficients  $b_n$ .

- (iii) If  $\theta_\infty = 0$ , the solution in case (ii) is not defined (denominators in the coefficients  $b_n$  vanish). We have anyway a solution equivalent to (17):

$$y(x) = 2 + \frac{\theta_0^2 - \theta_x^2 - 3}{6}x + ax^2 + \sum_{n=3}^{\infty} b_n(a; \theta_0, \theta_\infty)x^n.$$

If  $\theta_\infty = 2$ , we have a solution equivalent to (17). The parameter  $a$  is the coefficient of  $x^4$  (no parameter in lower powers of  $x$ ).

- (iv) If  $\theta_\infty = \frac{2n+1}{2}$ , the solution in (i) is not defined. Solutions exist equivalent to (16), provided that  $\theta_x \pm \theta_0$  is some integer.  
 (v) For  $\theta_\infty = -1$ , we have a solution equivalent to (17):

$$y(x) = 1 + ax^2 + \frac{a(\theta_x^2 - \theta_0^2 + 3)}{6}x^3 + \sum_{n=4}^{\infty} b_n(a; \theta_0, \theta_x)x^n.$$

- (vi) For  $\theta_\infty = 1$ , solutions of the desired form do not exist, except for  $y = 1$ .

We could proceed at our pleasure, choosing any value  $\theta_\infty \pm \theta_1$  integer. We would always have solutions of three kinds: (1) *one* out of the two solutions (15) and (18), (2) solutions equivalent to (16)—at least when  $\theta_\infty$  is half-integer—provided that  $\theta_x \pm \theta_0$  is some integer, (3) solutions equivalent to (17), for  $\theta_\infty$  equal to some integer.

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